Hall algebras and skein algebras, "in" Universität Wien

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(based on joint works with Morton and Pokorny)

I. Hall algebras
  (a) definition
  (b) the elliptic Hall algebra

II. Skein algebras
  (a) $\mathfrak{gl}_\infty$
  (b) $\mathfrak{sp}_\infty$

$\boxed{\text{I.}}$ The Hall algebra of an abelian category

(a) <u>Def</u>: Let $\mathcal{A}$ be an $\mathbb{F}_q$-linear abelian category with $\dim \operatorname{Hom}(M,N)$, $\dim \operatorname{Ext}^1(M,N) < \infty$ for all M, N.

Let $c^E_{M,N} = \# \left\{ E' \subset E \;\middle|\; \begin{matrix} E' \cong M \\ E/E' \cong N \end{matrix} \right\}$

The <u>Hall algebra</u> $H(\mathcal{A})$ is $\mathbb{Q}\{\text{iso classes of } Ob(\mathcal{A})\}$, with product

$$M \cdot N := \sum_E c^E_{M,N} E$$

<u>Exercise</u>: This is an associative product.

<u>Example</u>: $\mathcal{A}_n = \left\{ V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} V_n \;\middle|\; \begin{matrix} V_i \text{ an f.d. } \mathbb{F}_q \text{ v.s.} \\ f_i \text{ a linear map} \end{matrix} \right\}$ "type A_n quiver mod"

$M_{ij} = 0 \to 0 \to \mathbb{F}_q \xrightarrow{1} \mathbb{F}_q \to \cdots \to \mathbb{F}_q \to 0 \to \cdots$

$0 \to M_{22} \to M_{12} \to M_{11} \to 0$

$$M_{11} M_{22} = d[M_{22} \oplus M_{11}]$$
$$M_{22} M_{11} = c[M_{12}] + d[M_{22} \oplus M_{11}]$$

Thm: [Ringel, 90's] $H(A_n) \cong \mathcal{U}_{v=q}(sl_n^+)$,

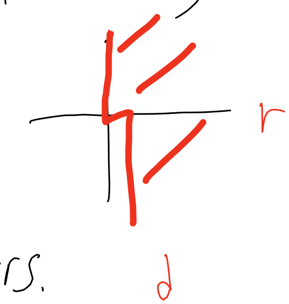
$M_{ij} \longmapsto$ Lusztig's PBW basis element (like elementary matrix E_{ij})

Rmk: $\mathcal{U}_v$ an algebra over $\mathbb{Q}[v^{\pm 1}]$, specialize $v = q (= \# \mathbb{F}_q)$ in theorem

(b) The Elliptic Hall algebra (of Burban and Schiffmann)

Def: Let $\mathcal{E}$ be the $\mathbb{C}[v^{\pm 1}, t^{\pm 1}]$-algebra generated by $\{u_{r,d} \mid r, d \in \mathbb{Z}\}$ subject to relations ......

Let $\mathcal{E}^+ \subset \mathcal{E}$ be generated by $\{u_{r,d} \mid r \geq 1 \text{ or } r = 0 \text{ and } d \geq 1\}$



Rmk: There is an $SL_2(\mathbb{Z})$ action which permutes the generators of $\mathcal{E}$.

Let X be a smooth elliptic curve over $\mathbb{F}_q$,

Thm: [Burban, Schiffman '05] $\exists\, \sigma(X), \bar{\sigma}(X) \in \mathbb{C}$ such that

$$\mathcal{E}^+_{v=\sigma(X),\, t=\bar{\sigma}(X)} \cong H^{sph}(Coh(X))$$

$u_{r,d} \longmapsto$ "a certain average of sheaves of rank r, degree d

<u>Technical note:</u> H^{sph} is the "spherical subalgebra," consisting of certain averages of sheaves (H is "too big")

<u>Opinion:</u> This is a remarkable theorem!

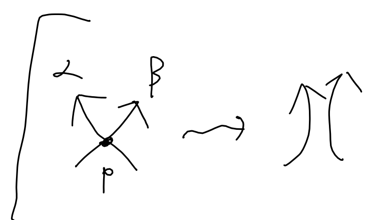
<u>Rmk:</u> $\mathcal{E}_{v,t}$ has since been related to many things (Hilbert schemes, symmetric functions, knot homology, ...

$\boxed{\text{II.}}$ <u>Skein algebras:</u> Quantization of character varieties of (topological) surfaces

(a) $\mathfrak{gl}_\infty$:

- $\mathrm{Char}(\Sigma) := \mathrm{Hom}_{Grp}(\pi_1(\Sigma), Gl_n) /\!/ Gl_n$; $\mathcal{O}\mathrm{char} := \mathcal{O}(\mathrm{Char}(\cdot\text{-}))$
- Goldman showed $\mathrm{Char}(\Sigma)$ is symplectic; $\{-,-\}: \mathcal{O}\mathrm{char} \otimes \mathcal{O}\mathrm{char} \to \mathcal{O}\mathrm{char}$ associated Poisson bracket
- If $\alpha: S^1 \to \Sigma$, $tr_\alpha \in \mathcal{O}\mathrm{char}$, $tr_\alpha(\rho) := tr(\rho(\alpha))$

$$(*) \quad \{tr_\alpha, tr_\beta\} = \sum_{p \in \alpha \cap \beta} \pm\, \alpha \#_p \beta \longleftarrow$$



$=: \alpha \#_p \beta$

<u>Rmk:</u> This doesn't depend on n. In fact, $(*)$ defines a Lie alg. $\mathcal{L}(\Sigma)$ and $\mathcal{L}(\Sigma) \twoheadrightarrow \mathcal{O}\mathrm{char}(n)$ is a Lie alg. map.

$\exists$ bialg. structure: $\alpha \mapsto \sum_{\substack{\text{resolutions} \\ \text{of self-crossing}}} \alpha_1 \otimes \alpha_2$

90's

Q: How to quantize? A: Turaev ...

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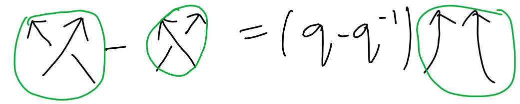
Def: The skein algebra $Sk_q(\Sigma)$ is

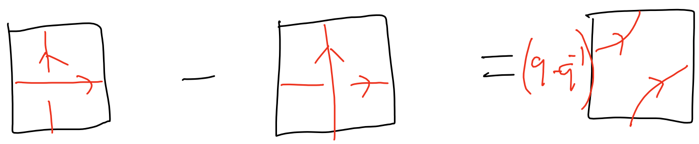
$$Sk_q(\Sigma) := \frac{\mathbb{Q}(q)\{\text{links in } \Sigma\times[0,1]\}}{\text{skein relation}}$$



$R - R^{-1} = q - q^{-1}$

- Multiplication is "stack in the $[0,1]$ direction"
- a "Link" is an embedded closed 1-manifold, up to ambient isotopy.
- $Sk_q(D^2) =$ ground ring
  Knot $\mapsto$ HOMFLYPT polynomial
- $Sk_q(\Sigma)$ graded by $H_1(\Sigma)$

Ex: If $\Sigma = T^2$,



$\Rightarrow [P_{1,0}, P_{0,1}] = (q-q^{-1})P_{1,1}$

Thm: [Morton, S. 17] $\exists\, P_x \in Sk_q(T^2)$ for $x \in \mathbb{Z}^2$ such that

$$Sk_q(T^2) \cong \frac{\langle P_x \mid x \in \mathbb{Z}^2\rangle}{P_xP_y - P_yP_x = (q^d - q^{-d})P_{x+y}}, \qquad d = \det[x\, y]$$

Rmk: # crossings on LHS $\approx d$, so would naively expect $\approx 2^d$ terms on RHS.

Rmk: 1) Simple closed curves are primitive $\Rightarrow P_x$ primitive $\forall x$

2) quantizing $\mathcal{U}(\mathcal{L}(T^2))$ and we get $\mathcal{U}(\mathcal{L}_q)$ (!)

3) $\mathcal{L}_q = \mathrm{Lie}(\langle X^{\pm 1}, Y^{\pm 1}\rangle / XY = qYX)$

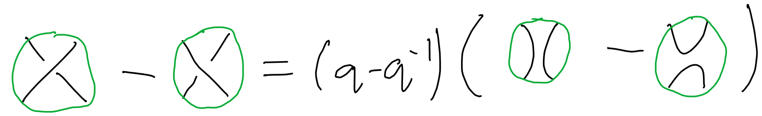
Cor: The skein algebra is isomorphic to the $v = t^{-1}$ specialization of the elliptic Hall algebra:

$$Sk_q(T^2) \cong \mathcal{E}_{q,q^{-1}}$$

The $SL_2(\mathbb{Z})$ actions agree.

(b) SO_∞ / Sp_∞ : Goldman's Poisson structure also quantizes.

$$\{tr_\alpha, tr_\beta\} = \sum_{p \in \alpha \cap \beta} \pm (\alpha \#_p \beta - \alpha \#'_p \beta)$$

Kauffman skein relations: 

Thm: [Morton, Pokorny, S. '19] The Kauffman skein algebra of the torus has the following presentation:

$$\frac{\mathbb{Q}(q)\langle D_x \mid x \in \mathbb{Z}^2 \rangle}{D_x D_y - D_y D_x = (q^d - q^{-d})(D_{x+y} - D_{x-y}), \quad D_x = D_{-x}}$$

Rmk: As before, a lot of cancellation in these relations

... cancellation in these relations.

$\boxed{\text{Question}}$ Is there a natural/geometric q,t-deformation of this algebra?

<u>Rmk:</u> $SK^{BCD}(T^2) \hookrightarrow SK^{A}(T^2)$

$D_x \longmapsto P_x + P_{-x}$

why?

Hecke alg $H_n \longrightarrow \mathrm{End}_{\mathcal{U}_q(\mathfrak{gl}_r)}(V^{\otimes n})$

BMW alg $BMW_n \longrightarrow \mathrm{End}_{BCD}(V^{\otimes n})$

Closure map

$$\begin{tikzcd} H_n \arrow[r] \arrow[d] & SK_q^{A}(\text{annulus}) \\ H_n/[H_n, H_n] \arrow[ur] & \end{tikzcd}$$

$$\begin{tikzcd} BMW_n \arrow[r] \arrow[d] & SK^{BCD}(\text{annulus}) \\ \cdot \arrow[ur] & \end{tikzcd}$$

$\hat{f}_n :=$ closure of symmetrizer

$$\sum_{k \geq 0} \frac{P_k}{k} z^k = \log\Big(1 + \sum_{j \geq 1} \hat{f}_j z^j\Big)$$

(same in both types) why?

$$\sum_{k \geq 0} \frac{D_k}{k} z^k = \log\Big(1 + \sum_{j \geq 1} \hat{f}_j z^j\Big)$$

$k \geq 0$

Mirror Symmetry: $\quad \underbrace{D^b(\mathrm{Coh}(X/\mathbb{C}))}_{} \cong \mathrm{Fuk}(T^2)$

elliptic curve

Cartoon: $\quad SK^A(T^2) \cong \mathrm{Ha}(\downarrow) \cong \mathrm{Ha}(\mathrm{Fuk}(T^2))$

$$SK^A(\Sigma) \overset{?}{\cong} \mathrm{Ha}(\mathrm{Fuk}(\Sigma))$$

some evidence [Cooper-S.], [Haiden]