Quot schemes & affine flag varieties

(Based on discussions w/ Bejleri, Garner, Gorsky, Oblomkov, Simental Rodriguez

Plan

1. Quot & Hilbert schemes on curves vs. affine Springer theory
2. Generalized affine Springer theory
3. Coherent - constructible correspondences

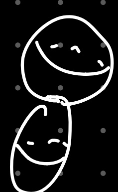
1. curve $X/\mathbb{C}$

<u>Def.</u> $\mathrm{Hilb}^n(X) = \{ Z \subset X \mid Z \text{ length } n \text{ subscheme} \}$

<u>E.g.</u>, If X smooth, then $\mathrm{Hilb}^n(X) = X^n/S_n$

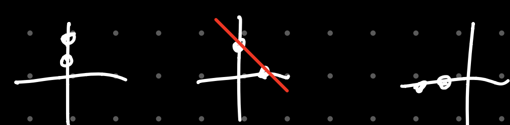
$\uparrow$ symmetric group

<u>E.g.</u>

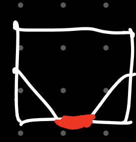


$\subset \mathbb{P}^2$

$\{xy=0\}$



$\mathrm{Hilb}^2(X)$ $\quad \mathbb{P}^2 \cup \mathrm{Bl}_{pt}(\mathbb{P}^1 \times \mathbb{P}^1) \cup \mathbb{P}^2$



Note: above $\{Z \subset X\} \leftrightarrow$

$$\{\mathcal{O}_X \twoheadrightarrow Q\} = \left\{ \mathcal{J}_Z \subset \mathcal{O}_X \;\middle|\; \dim_{\mathbb{C}} \mathcal{O}_X / \mathcal{J}_Z = n \right\}$$

If X is the <u>germ</u> of a singularity

e.g. $\dfrac{\mathbb{C}[[x,y]]}{f}$

If $f = xy$, $\mathcal{O}_X = \dfrac{\mathbb{C}[[x,y]]}{xy}$

$\mathrm{Hilb}^0 = pt$
$\mathrm{Hilb}^1 = pt$
$\mathrm{Hilb}^2 = \mathbb{P}^1$

$\uparrow$

$m^2 + \langle x + ay \rangle$

Could have taken

$\mathcal{F}$ any coherent shaf and looked at

$\mathrm{Quot}^n(\mathcal{F})$

For example, if C is Cohen-Macaulay and embedding dim ≤ 3,

$C \longrightarrow Y$ 3-fold
CY

PT invariants $PT_n(Y) = \left\{ \mathcal{O}_Y \xrightarrow{s} \mathcal{F} \;\middle|\; \begin{array}{l} \mathrm{coker}(s) \text{ length } n \\ \mathcal{F} \text{ pure dim 1 support} \end{array} \right\}$

Lemma (Bejleri–K.)

$$PT_n(C) = \{(\tilde{F}, s) \mid \operatorname{supp} \tilde{F} = C\}$$

$$\cong \operatorname{Quot}^n(\omega_C)$$

if C was planar (or Gorenstein)

this is $\cong \operatorname{Hilb}^n(C)$ (in original PT paper)

Rmk. $\operatorname{Quot}^n_C(\tilde{F})$ for torsion-free sheaves appears

in Yun's "Orbital integrals & Dedekind ζ-fns"

Yun proves $q^{?}(1-q)^{\#\text{branches}} \sum_{n\geq 0} \# \operatorname{Quot}^n(\omega_C)\, q^{ns}$

$C/\mathbb{F}_q$

$$\| $$

$$J_C(s)$$

satisfies a functional equation

Observation (Laumon–Ngô)

If C planar germ

$\mathcal{O}_C = \dfrac{\mathbb{C}[[x,t]]}{f}$ torsion-free sheaves of rank 1, ~~degree 0~~ are

$\mathcal{O}_C$-submodules

$\Lambda \subset \operatorname{Frac}(\mathcal{O}_C)$, $\Lambda \otimes_{\mathcal{O}_C} \operatorname{Frac}(\mathcal{O}_C) = \operatorname{Frac}(\mathcal{O}_C)$

identifying $\mathrm{Frac}(\mathcal{O}_C) = \underset{\displaystyle K}{\underset{\shortparallel}{\mathbb{C}((t))}}{}^{n}$ $\uparrow$ degree in x of f

the above becomes

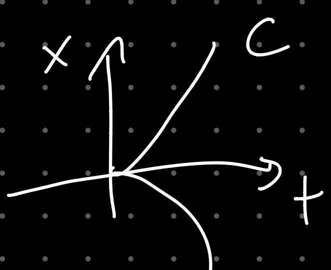
$$Sp_\gamma := \{ \Lambda \subset K^n \mid \Lambda \text{ stable under } \gamma = \text{companion matrix of } f \}$$

ex. $f = x^2 - t^2$

$\gamma = \begin{pmatrix} & t^2 \\ 1 & \end{pmatrix}$

affine Springer fibers for γ $\subset GL_n(K)/GL_n(\mathcal{O})$

$= Gr_{GL_n}$ affine Grassmannian



<u>Lemma</u> (Garner–K.)

$$\{ \Lambda \subset \mathcal{O}^n \mid \gamma\text{-stable lattice} \}$$

$$\cong \mathrm{Hilb}^\bullet(\mathrm{Spec}(\mathbb{C}[[t]][\gamma])) = \bigsqcup_{m \geq 0} \mathrm{Hilb}^m$$

$$Sp_\gamma = \{ gG(\mathcal{O}) \mid g^{-1}\gamma g \in \mathrm{Lie}(G(\mathcal{O})) \}$$

$(G = GL_n)$

Generalized affine Springer fibers

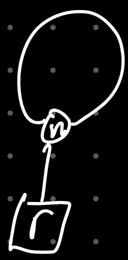
$N \in \mathrm{Rep}(G)$

$$G(K) \times N(\mathcal{O}) \xrightarrow{act} N(K) \ni v$$

$$\mathcal{M}_{G,N,v} := act^{-1}(v) / G(\mathcal{O})$$

Ex. If $N = Ad$, $v = \gamma \in \mathfrak{gl}_n((t))$

$$\mathcal{M}_{G,N,v} = Sp_\gamma$$



$N = Ad \oplus \mathbb{C}^n$ $\quad r = 1$

$$v = \gamma \oplus \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad \gamma = \begin{pmatrix} & & f \\ 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$

Thm (G-K)

$$\mathcal{M}_v \cong Hilb^\bullet(C)$$

Thm (K. - Simental Rodriguez) $\qquad f = f_1 \cdots f_r$, $\deg(f_i) = \lambda_i$

$$N = Ad \oplus \mathbb{C}^{nr}$$

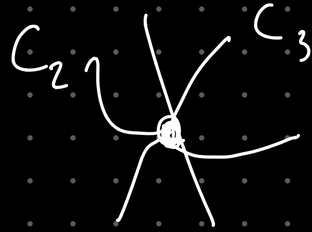
$$v = \left(\gamma_r = \begin{pmatrix} 1 \ddots 1 & f_1 & & \\ & 1 \ddots 1 & f_2 & \\ & & \ddots & \\ & & 1 \ddots 1 & f_r \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \right)$$

1 at the $\left(\sum_{i=1}^{j-1} \lambda_i\right) + 1$ th position

$$\mathcal{M}_v \cong \mathrm{Quot}^\bullet_C(\mathcal{O}_{\tilde C})$$

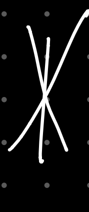
partial normalization where you separated branches

C



$\tilde C =$  C

$$\mathcal{O}_C \hookrightarrow \mathcal{O}_{\tilde C}$$

 ||| $\tilde C$

C

Note

This implies

"positive part"

$$\mathrm{Hilb}^\bullet(C) \cong \mathrm{Sp}_\gamma \cap \mathrm{Gr}^+_{GL_n}$$

$$\mathrm{Quot}^\bullet_C(\mathcal{O}_{\tilde C}) \cong \mathrm{Sp}_{\gamma_r} \cap \mathrm{Gr}^+_{GL_n}$$

$$\deg \Lambda = \dim(R / \Lambda \cap R) - \dim(\Lambda / R \cap \Lambda)$$

2. Springer theory

To (G, N), $N \in \mathrm{Rep}(G)$

Braverman - Finkelberg - Nakajima associate an algebra

$A_{G,N}$ "quantized Coulomb branch" (of 3d $N=4$ theory)

$:= H_*^{G(\mathcal{O}) \rtimes \mathbb{C}^*}(\underbrace{R_{G,N}}_{\text{space of triples}})$

space of triples $R_{G,N} = \{(g,s) \in G_K \times_{G_\mathcal{O}} N_\mathcal{O} \mid gs \in N_\mathcal{O}\}$

(analog of fiber of $St = \tilde{N} \times_N \tilde{N} \xrightarrow{\pi} G/B$)

Recall $Gr_G(\mathbb{C}) \simeq \Omega G$ $\quad H_*(\Omega G) \otimes H_*(\Omega G)$

$$\downarrow$$

"Pontryagin product" $H_*(\Omega G)$

$$Gr_G \times Gr_G \xleftarrow{p} G_K \times Gr \xrightarrow{q} G_K \times_{G_\mathcal{O}} Gr \xrightarrow{m} Gr$$

$(m \circ q)_* p^* : H_*^{G_\mathcal{O}}(Gr) \otimes H_*^{G_\mathcal{O}}(Gr) \to H_*^{G_\mathcal{O}}(Gr)$

associative (commutative) product

(studied by Bezrukavnikov–Finkelberg–Mirković

$\mathrm{Spec}(H_*^{G_\mathcal{O}}(Gr))$ = universal centralizer of $G^\vee$)

For $G_K \times_{G_\mathcal{O}} N_\mathcal{O}$, need to use

$\overset{\shortparallel}{T_{G,N}}$

$$\begin{tikzcd}
R_{G,N} \times R_{G,N} \arrow[d, hook] & p^{-1}(R_{G,N} \times R_{G,N}) \arrow[l, "p"'] \arrow[r, "q"] \arrow[d, hook] & q(\ldots) \to R_{G,N} \\
T_{G,N} \times R_{G,N} & G_K \times R_{G,N} \arrow[l, "p"'] &
\end{tikzcd}$$

Define p^* using "refined pullback"

then $(m \circ q)_* p^*$ defines an associative product on $A_{G,N} = H_*^{G_\mathcal{O} \rtimes \mathbb{C}^*}(R_{G,N})$

(commutative on $H_*^{G_\mathcal{O}}(R_{G,N})$, $\mathrm{Spec}(-) =$ "Coulomb branch")

<u>Want</u>:

$$A_{G,N} \otimes H_*^{\overset{\mathrm{Stab}(v)}{\overset{\|}{L_v}}}(M_v) \longrightarrow H_*^{L_v}(M_v)$$

<u>Thm</u> (G-K / Hilburn-Kamnitzer-Weekes)

$A_{G,N}$ acts on $H_*^{L_v}(M_v)$

(with assumptions on v)

Construction: $U_v := G_K \cdot v \cap N(\mathcal{O}) \subset N(K)$

$$R_{G,N} \times V_v \xleftarrow{p} p^{-1}(R_{G,N} \times V_v) \xrightarrow{q} q(\cdots) \xrightarrow{m} V_v$$

<u>Prop.</u> $\left[V_v / G_\mathcal{O}\right] \cong \left[L_v \backslash M_v\right]$

$$(m \circ q)_* p^* : H_*^{G_\mathcal{O}}(R_{G,N}) \otimes H_*^{G_\mathcal{O}}(V_v) \to H_*^{G_\mathcal{O}}(V_v)$$
$$= H_*^{L_v}(M_v)$$

<u>Thm</u> (Kodera–Nakajima)

If $N = Ad \oplus \mathbb{C}^{nr}$

$A_{G,N} \cong$ spherical cyclotomic RCA for $S_n \wr \mathbb{Z}/r\mathbb{Z}$

e.g. $r = 1$, $A_{G,N} =$ spherical RCA of S_n

$r = 0$ dDAHA

<u>Corollary1</u> (G.-U.) Spherical RCA $\circlearrowright$ $H_*^{\mathbb{C}^\times}(\mathrm{Hilb}^\bullet(\underset{\underset{C}{\cong}}{\mathrm{Spec}\,\mathbb{C}[[t]]}))$

<u>Corollary 2</u> –"– cyclotomic RCA

$$\circlearrowright H_*^{L_v}(\mathrm{Quot}_C^\bullet(\mathcal{O}_{\tilde{C}}))$$

In $\Gamma = 0$ case, known for many $v = \gamma$
that sph. ℓDAHA $\circlearrowright H_*^{\gamma}(Sp_\gamma)$
(classical affine Springer theory)

Thm above implies this is so for all regular γ
(s.s.)

<u>Note</u>. $A_{G,N}^{\hbar=0} \circlearrowright H_*^{?}(M_v)$

gives a sheaf (quasicoherent)
on $\mathrm{Spec}(A_{G,N}^{\hbar=0})$

More elaborate construction $\rightsquigarrow$
sheaf on $\mathrm{Proj}\left(\bigoplus_{d \geq 0} {}^{d}A_{G,N}^{\hbar=0}\right)$

$\gamma = t^d s$, $s \in \mathfrak{g}^{rs}(\mathbb{C})$

$$H_*^{T(\mathbb{C})}(\widetilde{Sp_\gamma}) \cong H^0(P \otimes P \otimes \mathcal{O}(d), \mathrm{Hilb}^n(\mathbb{C}^\times \times \mathbb{C}))$$

$$H_*^{T(\mathbb{C})}(\widetilde{M}_{\gamma n}) \cong H^0(P \otimes P \otimes \mathcal{O}(d), \mathrm{Hilb}^n(\mathbb{C}^2))$$