Attractor invariants, brane tilings and crystals

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Objects of study

1. Let X be an algebraic variety.
2. Assume that we want to count objects in $D^b(\operatorname{coh} X)$ having Chern character $\gamma \in H^*(X, \mathbb{Q})$.
3. "Counting" means: Construct the moduli space $M(\gamma)$ (or the moduli stack $\mathcal{M}(\gamma)$) of objects and consider an appropriate invariant: Euler number, Poincaré polynomial, motivic class, etc.
4. There can be too many objects: $\mathcal{O}(n) \oplus \mathcal{O}(-n)$ over $\mathbb{P}^1$ have the same Chern character for all $n \in \mathbb{Z}$. Therefore we need some restriction: "stability condition".
5. Given such stability condition Z, we consider the moduli space $M_Z(\gamma)$ of Z-semistable objects (or the moduli stack $\mathcal{M}_Z(\gamma)$) and take its invariant $[M_Z(\gamma)]$.
6. Then we define a generating function $\mathcal{A}_Z(x) = \sum_\gamma [\mathcal{M}_Z(\gamma)] x^\gamma$ and study its behavior under the variation of Z: "wall-crossing formulas".

Objects of study II

1. Assume that we have a direct sum of line bundles $T = \bigoplus_{i=1}^{r} L_i$ on X such that $\operatorname{Ext}^k(T, T) = 0$ for $k \geq 1$ and T generates $D^b(\operatorname{coh} X)$ (a tilting object).
2. Let $B = \operatorname{End}(T)^{\mathrm{op}}$. Then (under appropriate conditions) there is an equivalence

$$\Phi\colon D^b(\operatorname{coh} X) \to D^b(\operatorname{mod} B), \qquad F \mapsto \operatorname{RHom}(T, F).$$

3. This means that we can count objects in $D^b(\operatorname{mod} B)$ instead of counting them in $D^b(\operatorname{coh} X)$.
4. Usually we can describe B by a quiver with relations.
5. We are mostly interested in toric CY3-folds and they are usually encoded by quivers with potentials (or by brane tilings), which we discuss next.

Quivers with potentials and Jacobian algebras

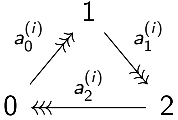
1. Let $Q = (Q_0, Q_1, s, t)$ be a quiver (finite directed graph), where $s, t\colon Q_1 \to Q_0$ are source and target maps. Let $\mathbb{C}Q$ be its path algebra.
2. We say that a path $p = a_n \dots a_1$ is a cycle if $t(a_n) = s(a_1)$.
3. Define a potential W in Q to be a linear combination of cycles. We use W to generate relations in the path algebra.
4. For any cycle $p = a_n \dots a_1$ and for any arrow $a \in Q_1$, define the (cyclic) derivation $\frac{\partial p}{\partial a} = \sum_{i\colon a_i = a} a_{i+1} \dots a_n a_1 \dots a_{i-1}$. We extend it to $\partial W / \partial a$ by linearity.
5. Define the Jacobian algebra $J = J(Q, W) = \mathbb{C}Q/(\partial W/\partial a : a \in Q_1)$.
6. Define a cut $I \subset Q_1$ to be a subset s.t. every term of W contains exactly one arrow from I.
7. Define $J_I = J_I(Q, W) = \mathbb{C}Q'/(\partial W/\partial a : a \in I)$, $Q' = Q \backslash I$.

Example: $\mathbb{C}^3$

1. Let Q be a quiver with one vertex 1 and three loops $x, y, z \colon 1 \to 1$.
2. Let $W = xyz - xzy$.
3. Then $\partial W / \partial z = xy - yx = [x, y]$ and similarly for other arrows.
4. Therefore $\mathbb{C}Q = \mathbb{C}\langle x, y, z \rangle$ and $J(Q, W) = \mathbb{C}[x, y, z]$.
5. Note that $J(Q, W)$ is the coordinate ring of a CY3-fold $\mathbb{C}^3$.
6. Usually in our examples $J(Q, W)$ will be non-commutative, but its center will be a coordinate ring of an (affine) singular 3CY variety.

Example: $\mathbb{P}^2$ and its canonical bundle

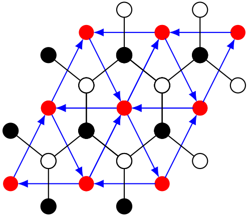
1. Consider projective plane $\mathbb{P}^2$ and the Beilinson exceptional collection $\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)$. Let $T = \mathcal{O} \oplus \mathcal{O}(1) \oplus \mathcal{O}(2)$.
2. Then $\mathrm{End}(T)$ corresponds to the quiver
$$0 \xrightarrow{a_0 \;\; b_0 \;\; c_0} 1 \xrightarrow{a_1 \;\; b_1 \;\; c_1} 2$$
with relations: $a_1 b_0 = b_1 a_0, \quad c_1 b_0 = b_1 c_0, \quad a_1 c_0 = c_1 a_0.$
3. We can extend it to the quiver



with potential $W = \sum_{i,j,k} \pm a_2^{(i)} a_1^{(j)} a_0^{(k)}$.
4. Then $\mathrm{End}(T) = J_I(Q, W)$ for $I = \left\{ a_2^{(1)}, a_2^{(2)}, a_2^{(3)} \right\}$.
5. Consider the canonical bundle $K_{\mathbb{P}^2}$ (a CY3-fold) and projection $\pi\colon K_{\mathbb{P}^2} \to \mathbb{P}^2$. Then $\mathrm{End}(\pi^* T) \simeq J(Q, W)$.
6. We have a diagram
$$\begin{array}{ccc} D^b(\mathrm{coh}\,\mathbb{P}^2) & \xrightarrow{\sim} & D^b(\mathrm{mod}\, J_I) \\ \big\downarrow & & \big\downarrow \\ D^b_c(\mathrm{coh}\, K_{\mathbb{P}^2}) & \xrightarrow{\sim} & D^b(\mathrm{mod}\, J) \end{array}$$

Brane tilings

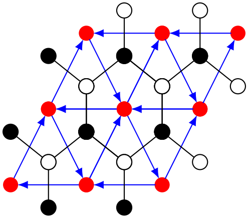
1. Brane tilings provide an important family of singular toric CY3-folds and their non-commutative crepant resolutions.
2. A brane tiling is a bipartite graph G embedded in a 2-dimensional (real) torus $\mathcal{T}$ – or periodic bipartite graph $\tilde{G}$ on $\mathbb{R}^2$.
3. We construct the quiver Q as the dual graph of G:
   3.1 The vertices $i \in Q_0$ correspond to faces of G (i.e. the connected components of $\mathcal{T} \setminus G$).
   3.2 The arrows $a\colon i \to j \in Q_1$ correspond to edges of G common to faces i and j.
   3.3 The arrows are oriented so that they go clockwise around white vertices of G and go anti-clockwise around black vertices of G.



Brane tilings II (Potential)

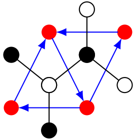
1. Let Q_2 be the set of connected components of $\mathcal{T} \backslash Q$, or equivalently, the set of vertices of G.
2. Let Q_2^+ and Q_2^- correspond to the sets of white and black vertices of G.
3. For any face $F \in Q_2$, let w_F be the cycle obtained by going along the arrows of F (defined up to a cyclic shift).
4. Define the potential W

$$W = \sum_{F \in Q_2^+} w_F - \sum_{F \in Q_2^-} w_F.$$



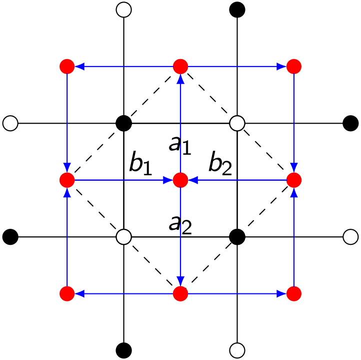
Example: $\mathbb{C}^3$

Consider the brane tiling and the quiver:



1. Red vertices are identified, and we get just one vertex in the quiver.
2. We get 3 arrows $x, y, z\colon 1 \to 1$.
3. Potential is $W = xyz - xzy$.

Example: Conifold



1. After identification we get vertices $1, 2$ and arrows $a_1, a_2 \colon 1 \to 2$, $b_1, b_2 \colon 2 \to 1$.
2. Potential is $W = a_1 b_1 a_2 b_2 - a_1 b_2 a_2 b_1$.
3. $\mathbb{C}Q/\partial W$ is non-commutative.
4. Its center $\simeq \mathbb{C}[x, y, u, v]/(xy - uv)$, $x \mapsto a_1 b_1 + b_1 a_1$, etc.
5. This is the coordinate ring of a conifold.

Example: $\mathbb{C}^3/\Gamma$

1. Let $\Gamma \subset \mathrm{SL}_3(\mathbb{C})$ be a finite abelian group.
2. Then $\mathbb{C}^3 \simeq \rho_1 \oplus \rho_2 \oplus \rho_3$, where ρ_i are 1-dimensional Γ-representations. We have $\rho_1\rho_2\rho_3 = 1$.
3. Consider the corresponding McKay quiver:
   3.1 Q_0 is the set of characters $\hat{\Gamma} = \mathrm{Hom}(\Gamma, \mathbb{C}^*)$.
   3.2 Arrows are $a_{i,\rho}\colon \rho\rho_i \to \rho$ for $\rho \in \hat{\Gamma}$, $i = 1, 2, 3$.
4. For any $\rho \in \hat{\Gamma}$ and $\pi \in \mathfrak{S}_3$, consider the cycle

$$\rho = \rho\rho_{\pi(1)}\rho_{\pi(2)}\rho_{\pi(3)} \to \rho\rho_{\pi(1)}\rho_{\pi(2)} \to \rho\rho_{\pi(1)} \to \rho$$

5. Define potential W to be a linear combination of all such cycles with appropriate signs.
6. We can glue the corresponding triangles to get a quiver embedded in a torus. Taking the dual graph we obtain a brane tiling.

Moduli spaces

1. Let (Q, W) be a quiver with potential (induced by a brane tiling).
2. For any dimension vector $d \in \mathbb{N}^{Q_0}$, consider the space of representations $R(Q, d) = \bigoplus_{a\colon i \to j} \operatorname{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j})$.
3. For $J = J(Q, W)$, we have a closed subset $R(J, d) \subset R(Q, d)$ of representations satisfying relations $\partial W / \partial a = 0$.
4. They are equipped with an action of the group $G_d = \prod_i \mathrm{GL}_{d_i}(\mathbb{C})$.
5. Consider a central charge $\mathrm{Z} = -\theta + \mathrm{i}\rho\colon \mathbb{Z}^{Q_0} \to \mathbb{C}$ with $\rho(e_i) > 0$.
6. For any representation M, define $d = \underline{\dim} M = (\dim M_i)_i \in \mathbb{N}^{Q_0}$ and $\mu_{\mathrm{Z}}(M) = \theta(d)/\rho(d)$.
7. A representation M of Q (or J) is called semistable if $\mu_{\mathrm{Z}}(N) \leq \mu_{\mathrm{Z}}(M)$ for any $N \subset M$.
8. Let $R_{\mathrm{Z}}(J, d) \subset R(J, d)$ be the subspace of semist. repr. and $M_Z(J, d) = R_{\mathrm{Z}}(J, d) /\!\!/ G_d$.

Invariants

1. We define "stacky" DT invariants (informally) by
$$\mathcal{A}(x) = \sum_{d\in\mathbb{N}^{Q_0}} \mathcal{A}(d,y)x^d = \sum_{d\in\mathbb{N}^{Q_0}} (-y)^{\chi_Q(d,d)} \frac{P(R(J,d))}{P(G_d)} x^d,$$
where χ_Q is the Euler form $\chi_Q(d,e) = \sum_i d_i e_i - \sum_{a\colon i\to j} d_i e_j$.
2. Here $P(X) = \sum_n \dim H^n(X)(-y)^n$ for smooth projective X and $P(X)$ is additive on complements.
3. Note that $\chi_Q(d,d) = \dim G_d - \dim R(Q,d)$.
4. In the presence of a cut $I \subset Q_1$, a rigorous formula is
$$\mathcal{A}(x) = \sum_{d\in\mathbb{N}^{Q_0}} (-y)^{\chi_Q(d,d)+2\gamma_I(d)} \frac{P(R(J_I,d))}{P(G_d)} x^d$$
where $\gamma_I(d) = \sum_{(a\colon i\to j)\in I} d_i d_j$.
5. For any stability function Z and ray $\ell \subset \mathbb{C}$, define
$$\mathcal{A}_{\mathsf{Z},\ell}(x) = \sum_{\mathsf{Z}(d)\in\ell} \mathcal{A}_{\mathsf{Z}}(d,y)x^d = \sum_{\mathsf{Z}(d)\in\ell} (-y)^{\chi_Q(d,d)} \frac{P(R_{\mathsf{Z}}(J,d))}{P(G_d)} x^d.$$

Wall-crossing formulas

1. Define the quantum affine space
$$\mathbb{A} = \bigoplus_{d \in \mathbb{N}^{Q_0}} \mathbb{Q}(y) x^d$$
equipped with the multiplication
$$x^d \circ x^{d'} = (-y)^{\langle d, d' \rangle} x^{d+d'}$$
where $\langle d, d' \rangle = \chi_Q(d, d') - \chi_Q(d', d)$ is a skew-symmetric form. Let $\hat{\mathbb{A}}$ be the completion of $\mathbb{A}$.
2. Basic wall-crossing formula: for any stability function Z, we have
$$\mathcal{A}(x) = \prod_{\ell}^{\curvearrowright} \mathcal{A}_{Z,\ell}(x),$$
where the product runs over rays ℓ in the upper half-plane ordered clockwise. In particular, the right hand side is independent of the stability function Z.
3. We can write the above formula as
$\mathcal{A}(\gamma, y) = \sum_{\substack{\gamma = \alpha_1 + \cdots + \alpha_n \\ \alpha_1 >_Z \cdots >_Z \alpha_n}} (-y)^{\sum_{i<j} \langle \alpha_i, \alpha_j \rangle} \prod_i \mathcal{A}_Z(\alpha_i, y)$
where we write $\alpha <_Z \beta$ if $\mu_Z(\alpha) < \mu_Z(\beta)$.

Wall-crossing formulas II

The previous formula implies that one can recursively determine $\mathcal{A}_Z(d,y)$ from $\mathcal{A}(d,y)$. Therefore if we know $\mathcal{A}_Z$ for some Z, then we can determine $\mathcal{A}_{Z'}$ for any Z′.

Theorem (Joyce-Reineke formula)

Let Z and Z′ be two stability functions. Given a tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of vectors in $\mathbb{N}^{Q_0} \setminus \{0\}$ and $1 \leq k < n$, define $\alpha_{\leq k} = \alpha_1 + \cdots + \alpha_k$, $\alpha_{>k} = \alpha_{k+1} + \cdots + \alpha_n$ and

$$s_k(\alpha) = \begin{cases} -1 & \alpha_k \leq_Z \alpha_{k+1} \text{ and } \alpha_{\leq k} >_{Z'} \alpha_{>k}, \\ 1 & \alpha_k >_Z \alpha_{k+1} \text{ and } \alpha_{\leq k} \leq_{Z'} \alpha_{>k}, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\mathcal{A}_{Z'}(\gamma, y) = \sum_{\gamma = \alpha_1 + \cdots + \alpha_n} \prod_{k=1}^{n-1} s_k(\alpha) \cdot (-y)^{\sum_{i<j} \langle \alpha_i, \alpha_j \rangle} \cdot \prod_i \mathcal{A}_Z(\alpha_i, y).$$

Plethystic exponential

1. Later in the formulation of explicit results the notion of a plethystic exponential will be useful.
2. It is defined on $\hat{\mathbb{A}}^+$ (the maximal ideal of $\hat{\mathbb{A}}$) by $\operatorname{Exp}(f+g) = \operatorname{Exp}(f)\operatorname{Exp}(g)$ and on monomials by $\operatorname{Exp}(y^k x^d) = \sum_{m\geq 0} y^{mk} x^{md}$.
3. Explicitly $\operatorname{Exp}(f) = \exp\left(\sum_{m\geq 1}\frac{1}{m} f(y^m, x_1^m, \dots, x_n^m)\right)$.
4. Its inverse is the *plethystic logarithm* $\operatorname{Log}(f) = \sum_{m\geq 1}\frac{\mu(m)}{m}\log\left(f(y^m, x_1^m, \dots, x_n^m)\right)$, where μ is the Möbius function.
5. For example, let Q be a quiver with one vertex and no arrows.

$$\mathcal{A}(x) = \sum_{d\in\mathbb{N}} \frac{(-y)^{d^2}}{P(\mathrm{GL}_d)} x^d = \sum_{d\in\mathbb{N}} \frac{(-y)^{-d^2}}{(y^{-2})_d} x^d = \operatorname{Exp}\left(\frac{x}{y^{-1}-y}\right)$$

where $(q)_n = \prod_{i=1}^n (1-q^i)$ and $P(\mathrm{GL}_n) = q^{n^2}(q^{-1})_n$ with $q = y^2$. The above expression is called the *quantum dilogarithm*.

Other types of invariants

1. Assume that Z is generic, meaning that $\mu_Z(d) = \mu_Z(d')$ implies that $d \parallel d'$.
2. Define rational DT invariants $\bar{\Omega}_Z(d, y)$ by
$$\sum_{Z(d)\in\ell} \mathcal{A}_Z(d, y)x^d = \exp\left(\frac{\sum_{Z(d)\in\ell} \bar{\Omega}_Z(d, y)x^d}{y^{-1} - y}\right)$$
3. Define integer DT invariants $\Omega_Z(d, y)$ by
$$\sum_{Z(d)\in\ell} \mathcal{A}_Z(d, y)x^d = \mathrm{Exp}\left(\frac{\sum_{Z(d)\in\ell} \Omega_Z(d, y)x^d}{y^{-1} - y}\right)$$
4. Note that if Q is a symmetric quiver, meaning that $\chi_Q(d, d') = \chi_Q(d', d)$, then $\mathbb{A}$ is commutative and $\bar{\Omega}_Z(d)$, $\Omega_Z(d)$ are independent of Z.
5. If Q is not symmetric, then one can study invariants for a special stability parameter, called attractor stability.

Attractor invariants

1. Given a dimension vector $d \in \mathbb{N}^{Q_0}$, consider $\theta = \langle -, d \rangle : \mathbb{Z}^{Q_0} \to \mathbb{R}$ and let θ' be its generic perturbation. Define attractor invariants (also called initial data)
$$\bar{\Omega}_*(d, y) = \bar{\Omega}_{\theta'}(d, y)$$
and similarly for $\Omega_*(d, y)$ and $\mathcal{A}_*(d, y)$.
2. Theorem: Attractor invariants are independent of the perturbation.
3. Theorem: If Q is acyclic, then $\Omega_*(d) = 1$ for $d = e_i$ and zero otherwise.
4. One can see from the wall-crossing formulas that for any stability parameter Z, invariants $\bar{\Omega}_Z(d, y)$ can be recursively expressed in terms of attractor invariants.
5. Alexandrov and Pioline conjectured several explicit formulas (flow tree formula and attractor tree formula) that express $\bar{\Omega}_Z$ in terms of $\bar{\Omega}_*$. Both of them are proved now.
6. This means that knowing attractor invariants, we can determine DT invariants in all stability chambers.

Example: $\mathbb{C}^3$

Theorem (Behrend-Bryan-Szendroi)

$$\mathcal{A}(x) = \operatorname{Exp}\left(\frac{-y^3 \sum_{n\geq 1} x^n}{y^{-1} - y}\right).$$

Therefore $\Omega(n, y) = -y^3$.

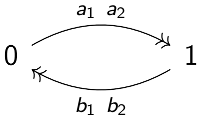
More generally, for any CY3-fold X the generating function counting D0 invariants (corresponding to coherent sheaves with 0-dimensional support) is

$$\mathcal{A}_{X,0}(x) = \operatorname{Exp}\left(\frac{(-y)^{-3} P(X) \sum_{n\geq 1} x^n}{y^{-1} - y}\right)$$

Under the equivalence $D^b_c(\operatorname{coh} X) \simeq D^b(\operatorname{mod} J)$ for $J = J(Q, W)$, the class of a point on X is mapped to the dimension vector $\delta = (1, \ldots, 1) \in \mathbb{Z}^{Q_0}$. Therefore

$$\Omega_*(n\delta) = (-y)^{-3} P(X).$$

Example: Conifold

Consider the quiver 

with potential $W = a_1 b_1 a_2 b_2 - a_1 b_2 a_2 b_1$. Note that this quiver is symmetric.

Theorem (MMNS)

$$\mathcal{A}(x) = \mathrm{Exp}\left(\frac{(q+q^2)x_0 x_1 - q^{\frac{1}{2}}(x_0 + x_1)}{q-1} \sum_{k\geq 0} x^{k\delta} \right).$$

where $q = y^2$. Therefore the only non-zero unframed DT invariants are

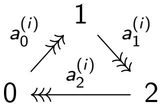
$$\Omega(n\delta) = -y^3 - y \ , \quad \Omega(n\delta - e_0) = \Omega(n\delta - e_1) = 1 \quad (n \geq 1)$$

Non-symmetric quivers

1. In the case of non-symmetric quivers we can not expect to obtain closed formulas for $\mathcal{A}(x)$.
2. DT invariants $\Omega_Z(d)$ and $\bar{\Omega}_Z(d)$ depend on Z.
3. Therefore now we are looking for attractor invariants $\Omega_*(d)$ (instead of $\Omega(d)$ in the symmetric case).
4. In order to compute attractor invariants, we compute first invariants $\mathcal{A}(d, y)$ and then apply wall-crossing formulas.
5. For the computation of $\mathcal{A}(d, y)$ we need to find motivic invariants of the representation spaces $R(J_I, d)$.
6. This is done using "dimensional reduction": assume that we have a second cut $I' \subset Q_1$ disjoint from I. Then we have a forgetful map $\pi\colon R(J_I, d) \to R(Q'', d)$, $Q'' = Q \backslash (I \cup I')$ with linear fibers.
7. If Q'' is a particularly simple quiver, then we can classify its representations and perform all the required computations.

Example: $\mathbb{C}^3/\mathbb{Z}_3$

1. Consider the action of $\mathbb{Z}_3$ on $\mathbb{C}^3$ given by $1 \mapsto \mathrm{diag}(\omega, \omega, \omega)$, $\omega = e^{2\pi \mathrm{i}/3}$.
2. The corresponding McKay quiver with potential is



with $W = \sum_{i,j,k} \pm a_2^{(i)} a_1^{(j)} a_0^{(k)}$.

3. We encountered it in our discussion of $\mathbb{P}^2$ and $K_{\mathbb{P}^2}$. Therefore our computation of attractor invariants for $J(Q, W)$ can be used to compute DT invariants on $\mathbb{P}^2$ and $K_{\mathbb{P}^2}$.

Conjecture

$$\Omega_*(e_i) = 1, \qquad \Omega_*(n\delta) = -y^{-1}(y^4 + y^2 + 1)$$

and all other invariants vanish.

This statement (except for the value of $\Omega_*(n\delta)$) was conjectured by Beaujard, Manschot, Pioline [2004.14466]. It was the main motivation for our study of attractor invariants of brane tilings.

Example $\mathbb{C}^3/\mathbb{Z}_N$

1. More generally, consider the action of $\mathbb{Z}_N$ on $\mathbb{C}^3$ given by $1 \mapsto \operatorname{diag}(\omega, \omega, \omega^{-2})$, $\omega = e^{2\pi \mathrm{i}/N}$.
2. The corresponding singular 3CY is $\mathbb{C}^3/\mathbb{Z}_N$ and its crepant resolution $\widetilde{X}$ has Poincaré polynomial ($q = y^2$)

$$P(\widetilde{X}) = \begin{cases} q(q^2 + kq + k - 1) & N = 2k \\ q(q^2 + kq + k) & N = 2k + 1 \end{cases}$$

Conjecture

$$\Omega_*(e_i) = 1, \qquad \Omega_*(n\delta) = (-y)^{-3} P(\widetilde{X}),$$
$$\Omega_*(e_i + e_{i+2} + \cdots + e_{i-2} + n\delta) = -y, \qquad n \geq 1, \ \textit{even } N.$$

All other attractor invariants vanish.

Framed (NCDT) invariants

1. Refined invariants in the above conjectures were obtained using a direct computation.
2. We can also test them by computing unrefined framed invariants using different methods
3. For any framing vector f, let Q^{f} be a quiver obtained from Q by adding a new vertex ∞ and f_i arrows $\infty \to i$, for $i \in Q_0$.
4. Let $J^{\mathsf{f}} = J(Q^{\mathsf{f}}, W)$, $d^{\mathsf{f}} = (d, 1)$ and $R^{\mathsf{f}}(J, d) = R(J^{\mathsf{f}}, d^{\mathsf{f}})$.
5. Define $R^{\mathsf{f},\mathrm{NC}}(J, d) \subset R^{\mathsf{f}}(J, d)$ to be the subspace of representations M such that if $N \subset M$, $N_\infty \neq 0$, then $N = M$ (we call such M NC-stable; they are generated by M_∞). Define $M^{\mathsf{f},\mathrm{NC}}(J, d) = R^{\mathsf{f},\mathrm{NC}}(J, d)/G_d$.
6. We define the generating function of NCDT invariants $Z_{\mathrm{NC}}(x) = \sum_d (-1)^{\chi_Q(d,d) - \mathsf{f}\cdot d} e(M^{f,\mathrm{NC}}(J, d)) x^d$.
7. One can obtain framed invariants from unframed invariants. For symmetric quivers the formula is $Z_{\mathsf{f},\mathrm{NC}}(x) = \bar{S}_{\mathsf{f}} \operatorname{Exp}\left(-\sum_d (\mathsf{f}\cdot d)\Omega(d, 1)x^d\right)$, $\bar{S}_{\mathsf{f}}(x^d) = (-1)^{\mathsf{f}\cdot d} x^d$.

NCDT invariants and crystals

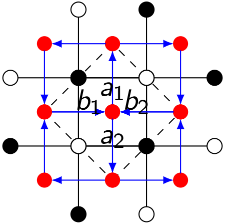
1. On the other hand, we can compute NCDT invariants for $f = e_i$ using molten crystals.
2. Let Δ_i denote the set of paths starting at i up to equivalence, where two paths are equivalent if they are equal in J.
3. There is a partial order on Δ_i, where $u \leq v$ if $v \sim wu$ for some path w. One defines an ideal to be a (finite) subset $\mathcal{I} \subset \Delta_i$ such that $u \leq v$ and $v \in \mathcal{I}$ implies $u \in \mathcal{I}$.
4. For example for $J(Q, W) = \mathbb{C}[x, y, z]$ one can identify Δ_1 with $\mathbb{N}^3$. Then $\mathcal{I} \subset \mathbb{N}^3$ is an ideal iff $(k, l, m) \in \mathcal{I}$ and $k' \leq k, l' \leq l, m' \leq m$ imply $(k', l', m') \in \mathcal{I}$. Such subsets are also known as plane partitions.
5. One defines a molten crystal to be the complement of a (finite) ideal.
6. Theorem:

$$Z_{e_i,\mathrm{NC}}(x) = \sum_{\mathcal{I} \subset \Delta_i, d = \underline{\dim} \mathcal{I}} (-1)^{\chi_Q(d,d) + d_i} x^d$$

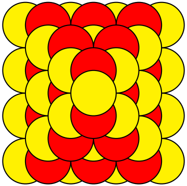
$\underline{\dim}\mathcal{I} = \sum_{u \in \mathcal{I}} e_{t(u)} \in \mathbb{Z}^{Q_0}$

Example

For example, for the conifold



the pyramid Δ_1 has the form



and counting its ideals we can determine $Z_{e_1,\mathrm{NC}}(x)$. The same method can be applied to other brane tilings and we can test our conjectures for attractor invariants $\Omega_*(d)$.