

Representing Hochschild cohomology

of Soergel bimodules

\$Bim\$

$$g^* \circ h^* = sl_n, W \cong S_n \subset h^*$$

$$S = \{(12), (23), \dots, (n-1n)\}$$

$$R = k[h^*], \text{ char } k = 0, k = \bar{k} \quad (k = \mathbb{Q}_\ell)$$

$$= \text{Sym}[h^*]$$

$$\deg h = 2$$

$R\text{-mod-}R$ - category of graded R -bimodules

Monoidal category, \otimes_R -monoidal str.

$\$Bim_n \subset R\text{-mod-}R$ full subcat., closed under summands, \otimes_R , grading shifts, containing $B_s := R \otimes_R R(1)$.

Isom. classes of indec. $\leadsto B_w, w \in W$

Rouquier complexes: $Br_n = Br(W, S)$ braid group on n -strands.

$$b \in Br_n \leadsto F_b \in \text{Ho}(\$Bim_n)$$

$$b_s, s \in S \mapsto \Delta_s = B_s \rightarrow R(1)$$

Hochschild cohomology:

$$M \in R\text{-mod-}R$$

$$HH^*(M) = \text{Ext}_{R\text{-mod-}R}^i(R, M)$$

Goal: find $X_k \in \text{Ho}(\$Bim_n)$

$$\text{for } M \in \$Bim_n, HH^k(M) = \text{Hom}_{\text{Ho}(\$Bim_n)}(X_k, M)$$

Example 1: $X_0 = R$

Example 2: $X_{\text{top}} = X_{n-1} = F_{w_0}^{\otimes 2} = \text{Full Twist}$

Gorsky - Hogancamp - Mellit - Nakagane

Motivation: Khovanov - Rozansky homology:

L - link in S^3 , $L = \bar{\beta}$, $\beta \in Br_n$.

$$HHH^k(L) = HH^k(F_{\bar{\beta}}) \stackrel{\text{def}}{=} H^*(\dots \rightarrow HH^k(F_{\bar{\beta}}^n) \rightarrow HH^k(F_{\bar{\beta}}^{n+1}) \rightarrow \dots)$$

[GHMN] result above categorifies a result of Kalman, relating coef's of HOMFLY - PT polynomials of $\bar{\beta}$ & $w_0^2 \bar{\beta}$.

First answer:

$HH^*(M)$ can be computed using the Koszul resolution K^\bullet of $R \in R\text{-mod-}R$

K^\bullet is a complex of free $R \otimes R$ -modules

$$HH^*(M) = H^*(\text{Hom}(K^\bullet, M))$$

complex of Hom's

Fact: $M \in \$Bim_n$, $R \otimes R$ -action factors through $R \otimes_{R \rtimes W} R$ -action.

$$K'_s := K^\bullet \otimes_{R \otimes R} R \otimes_{R \rtimes W} R$$

Each term of K'_s is of the form $B_{w_0}^{\otimes k_i}(r_i)$ ($B_{w_0} \cong R \otimes_{R \rtimes W} R(l_{w_0})$)

Theorem 1:

there is a t -structure on $\text{Ho}(\$Bim_n)$ w. the cohomology functor ${}^p\mathcal{H}_{w_0}^\bullet$, such that $HH^k(M) \cong \text{Hom}({}^p\mathcal{H}_{w_0}^{-k}(K'_s), M)$.

In particular, ${}^p\mathcal{H}_{w_0}^0(K'_s) \cong R$.

(\cong - up to grading shift) ${}^p\mathcal{H}_{w_0}^{\text{top}}(K'_s) \cong F_{w_0}^{\otimes 2}$.

Geometry:

G -reductive / \mathbb{F}_q , $B \subset G$ - Borel (split)

$U \subset B$ - unipotent radical T - maximal torus.

$\text{Ho}(\$Bim_n)$ - "algebraic Hecke category"

Webster-Williamson geom. int. of HHH

$$D_{\text{mix}}^b(B \backslash G/B) \simeq_{\text{[Bezrukavnikov-Yun]}} D_{\text{mix, mon}}^b(\cup G/U)$$

(unip.) monodromic = "locally-constant" along fibers of $G/U \twoheadrightarrow G/B$

completed = allow free-monodromic unipotent local systems

Example: $e = eB \in G/B$, 0-dim. B -orbit.

$$T \simeq B/U \simeq \pi^{-1}(e)$$

$$R \in \$Bim_n$$

$$\mathcal{S}_e \in D_{\text{mix}}^b(\cup G/B)$$

$$\hat{L} \in D_{\text{mix, mon}}^b(\cup G/U)$$

\hat{L} - pro-local system on T corresp. to "infinite Jordan block".

$$B_w, \Delta_w, \nabla_w$$

$$IC_w, j_{w*} \bar{Q}_e, j_{w*} Q_e \quad \tilde{T}_w, \tilde{j}_{w*} \hat{L}, \tilde{j}_{w*} \hat{\nabla}_w$$

$$T \times U_w \simeq \hat{G}_w \xrightarrow{\hat{j}_w} G/U \quad \tilde{\Delta}_w, \tilde{\nabla}_w$$

$$\text{Bruhat cell } U_w \xrightarrow{j_w} G/B$$

Both categories are monoidal via group convolution $*$.

\tilde{T}_{w_0} - "big projective".

$$\cup G/U \xrightarrow{\mathbb{P}} \frac{\cup G/U}{T} \leftarrow \text{adjoint action}$$

Theorem 2

$$1) K'_s \leadsto p^* p_! \tilde{T}_{w_0}$$

$$2) {}^p\mathcal{H}_{w_0}^\bullet(M) \leadsto \hat{\Delta}_{w_0} * {}^p\mathcal{H}^*(\hat{G}_w * \tilde{M})$$

cohomology w.r.t. shifted perverse t -structure.

$$3) \tilde{T}_w \text{ are injective w.r.t. this } t\text{-str.}$$

$$H^*(\text{Hom}(K'_s, \tilde{T}_w)) \simeq \text{Hom}({}^p\mathcal{H}_{w_0}^{-k}(p^* p_! \tilde{T}_{w_0}), \tilde{T}_w)$$

$$X_k \leadsto {}^p\mathcal{H}_{w_0}^{-k}(p^* p_! \tilde{T}_{w_0})$$

Second answer (computation of X_k).

$$D^b(\cup G/U)$$

$$D^b(G/G) \xrightarrow{Av_U!} D^b(\frac{\cup G/U}{T})$$

$Av_U^{-1}(D_{\text{mon}}^b)$ = derived category of character sheaves.

DCS.

$D^b(G/G)$ is a monoidal category w.r.t. to the $!$ -group convolution $*$

$Av_U!$ is a monoidal functor.

$$\hat{S}_G \in \text{pro DCS}, {}^* Av_U!(\hat{S}_G) = \tilde{\Delta}_e = \tilde{\nabla}_e$$

(pro)unit in DCS

Key fact: $Av_U!$ restricted to DCS, is t -exact

perverse t -str. $\mapsto w_0$ -shifted perverse t -str.

$\mathcal{N}^v \hookrightarrow G$ - unipotent variety.

$j: \mathcal{N}^{\text{reg}} \hookrightarrow \mathcal{N}^v$ - regular orbit.

Theorem 3

$$Av_U!(j_* \bar{Q}_e * \hat{S}_G) \cong p_! \tilde{T}_{w_0}$$

Corollary 1:

$${}^p\mathcal{H}_{w_0}^\bullet(p^* p_! \tilde{T}_{w_0}) \cong p^* Av_U!({}^p\mathcal{H}^*(j_* \bar{Q}_e) * \hat{S}_G)$$

Theorem 4 [$G = PGL_n$]

$${}^p\mathcal{H}^k(j_* \bar{Q}_e) \cong \text{Spr}_{\lambda^{\text{ark}}} - \text{summand}$$

of the Springer sheaf corresp. to the exterior powers of the reflection rep. of $W = S_n$

Corollary 2:

$$1) {}^p\mathcal{H}_{w_0}^0(K'_s) \cong R$$

$$2) {}^p\mathcal{H}_{w_0}^{-n+1}(K'_s) \cong F_{w_0}^{\otimes 2}$$

3) ${}^p\mathcal{H}_0^k(K'_s)$ has a filtration by braid objects, categorifying the elem. symmetric polynomials in Jucys-Murphy braids.

+ some shifts & Take twists