Cuspidal cohomology for 2CY categories

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The setup

$\mathcal{A}$ will be a (left) 2CY category, closed under extensions, with stack of objects $\mathfrak{M}_{\mathcal{A}}$ and morphism $\mathtt{JH}_{\mathcal{A}} \colon \mathfrak{M}_{\mathcal{A}} \to \mathcal{M}_{\mathcal{A}}$ to its (good) coarse moduli space. Running examples of $\mathcal{A}$:

1. The subcategory of ζ-semistable coherent sheaves with fixed slope and compact support on a smooth complex quasiprojective surface S with $\mathcal{O}_S \cong \omega_S$, or the category of semistable Higgs sheaves of fixed slope on a smooth complex projective curve C.
2. The category of finite-dimensional representations (or semistable f.d. representations of fixed slope) for Π_Q (a preprojective algebra) or Λ_Q^q (a multiplicative preprojective algebra) or $\mathbb{C}[\pi_1(\Sigma_g)]$ (a fundamental group algebra).

Today's goal

Our goal is to understand the Hall algebra structure on $\mathcal{H}_{\mathcal{A}} = \mathrm{H}^{\mathrm{BM}}(\mathfrak{M}_{\mathcal{A}}, \tilde{\mathbb{Q}})$, the (shifted) Borel–Moore homology of the stack of objects in $\mathcal{A}$, in particular the generators of the BPS Lie algebra $\mathfrak{g}_{\mathcal{A}} \subset \mathcal{H}_{\mathcal{A}}$.

Opening remarks

Generalizations

- We can let $\mathcal{S} \subset \mathcal{A}$ be a full Serre subcategory of our 2CY category, same structure theorems as for $\mathcal{H}_{\mathcal{A}}$ hold for $\mathcal{H}_{\mathcal{S}}$.
- In the surface examples we can consider not-necessarily semistable sheaves. Same results hold, with more work.
- If $\mathbb{C}^*$ acts on $\mathcal{A}$, can consider $\mathbb{C}^*$-equivariant BM homology, a flat deformation of $\mathcal{H}_{\mathcal{A}}$ if $\mathcal{H}_{\mathcal{A}}$ pure.

The Lie algebra $\mathfrak{g}_{\mathcal{A}}$

- The BPS Lie algebra $\mathfrak{g}_{\mathcal{A}}$ has two definitions, both involving the "third dimension".
- Think of $\mathfrak{g}_{\mathcal{A}}$ as generalised KM Lie algebra (in both senses of "generalised KM Lie algebra"). In the context of (the first part of) this talk, it is easier to define $\mathrm{U}(\mathfrak{g}_{\mathcal{A}})$ than $\mathfrak{g}_{\mathcal{A}}$. The question of generators is the same though.

General background: DT theory, GRT, geometry

- Want to understand the BM homology the stack $\mathfrak{M}_{\mathcal{A}}$, along with Hodge structures and associated Hall algebras/geometric representation theory.
- The point of view we use to understand this homology is that it is really the vanishing cycle cohomology of the stack of objects in an associated 3CY completion, and thus is amenable to cohomological DT theory.
- This "extra dimension" helps prove many foundational theorems regarding the BM homology/Hall algebra of our 2CY categories. For example
  1. "Integrality", which states that there is an equality $[\bigoplus_{\gamma\in\mathsf{K}(\mathcal{A})}\mathsf{H}^{\mathsf{BM}}(\mathfrak{M}_{\gamma}(\mathcal{A}))]=[\mathsf{Sym}(\bigoplus_{\gamma}\mathsf{BPS}_{\mathcal{A},\gamma}\otimes\mathsf{H}(\mathrm{pt}/\mathbb{C}^*))]$ where each $[\mathsf{BPS}_{\mathcal{A},\gamma}]$ is class of something finite-dimensional.
  2. Local and global purity theorems, that enable us to calculate the size/generators of $\mathfrak{g}_{\mathcal{A}}$ (and define it in the first place!).
  3. The construction of a cocommutative coproduct.

Background: Hodge theory

The Borel–Moore homology $\mathrm{H}^{\mathrm{BM}}(\mathfrak{M},\mathbb{Q})$ carries a mixed Hodge structure, and in particular there is a weight filtration $W_\bullet \mathrm{H}_i^{\mathrm{BM}}(\mathfrak{M},\mathbb{Q})$.

- We define $\chi_q(\mathfrak{M})=\sum_{i,n\in\mathbb{Z}}(-1)^i\dim(\mathrm{Gr}_n^W \mathrm{H}_i^{\mathrm{BM}}(\mathfrak{M},\mathbb{Q}))q^{n/2}\in \mathrm{K}_0(\mathrm{Vect}_{\mathbb{Z}})$
- By definition $\mathfrak{M}$ is **pure** if $\chi_q(\mathfrak{M})$ agrees with the (compactly supported) Poincaré series of $\mathfrak{M}$, i.e. $\mathrm{Gr}_n^W \mathrm{H}_i^{\mathrm{BM}}(\mathfrak{M},\mathbb{Q})=0$ for all $i\neq n$.

E.g.

$$\chi_q(\mathbb{C}^*)=q-1\neq q-q^{1/2}=p(\mathbb{C}^*)$$
$$\chi_q(\mathbb{P}^1)=q+1=p(\mathbb{P}^1)$$

so $\mathbb{C}^*$ is *not* pure, and $\mathbb{P}^1$ is.

- Purity of the MHM complex $\mathcal{L}=\mathtt{JH}_!\,\mathbb{Q}_{\mathfrak{M}_{\mathcal{A}}}$ allows for a similar move: working out what $\mathcal{L}$ *is* from its *class* in $\mathrm{K}_0(\mathrm{MHM}(\mathcal{M}_{\mathcal{A}}))$
- Purity enables us to go from point counting/motivic calculations to cohomology.

The geometric motivation: Halpern–Leistner's conjecture

Conjecture (Halpern–Leistner 2015)

Let S be a K3 surface, i.e. S is a smooth projective surface with $\mathcal{O}_S \cong \omega_S$ and $\mathrm{H}^1(S, \mathcal{O}_X) = 0$. Fix a polarization ζ, and let $\mathfrak{M}_\nu^{\zeta\text{-sst}}(S)$ be the moduli stack of Geiseker semistable coherent sheaves with Mukai vector ν. Then

$$\mathcal{H}_{\mathcal{A},\nu} := \mathrm{H}^{\mathrm{BM}}(\mathfrak{M}_\nu^{\zeta\text{-sst}}(S), \mathbb{Q})$$

is pure.

- The conjecture says that the BM homology of stacks of sheaves on K3 surfaces is *motivic*: i.e. we can calculate it via "cut and paste" and/or point counting over finite fields.
- Assume $\mathcal{L} = \mathtt{JH}_{S,!}\,\mathbb{Q}_{\mathfrak{M}_\nu^{\zeta\text{-sst}}(S)}$ is a pure complex of MHMs. The hypercohomology of a pure complex of MHMs on projective space is pure, so the conjecture follows from projectivity of $\mathcal{M}_\nu^{\zeta\text{-sst}}(S)$.
- Purity of $\mathcal{L}$ (as opposed to $\mathcal{H}_{\mathcal{A}}$) is much more useful from the GRT point of view (in particular: constructing generators).

Formality

Let $\mathcal{A}$ be a left 2CY dg category (e.g. dg enhancement of derived category of any of our running examples). Call an object $\mathcal{F} \in \mathrm{Ob}(\mathcal{A})$ a **Σ-object** if

$$\mathrm{Ext}^{\bullet}(\mathcal{F}, \mathcal{F}) \cong \mathrm{H}^{\bullet}(\Sigma_g, \mathbb{C})$$

for some compact Riemann surface Σ_g. Call a set of Σ-objects $\{\mathcal{F}_1, \ldots, \mathcal{F}_n\}$ in $\mathcal{A}$ a Σ-collection if $\mathrm{Ext}^{\bullet}(\mathcal{F}_i, \mathcal{F}_j)$ is concentrated in degree 1 if $i \neq j$.

Theorem

Let $S = \{\mathcal{F}_1, \ldots, \mathcal{F}_n\}$ be a Σ-collection in $\mathcal{A}$. Then the full subcategory containing S is formal.

Result builds on work of Brav-Dyckerhoff and Ren, generalising results of e.g. Deligne, Simpson, Van den Bergh, Kaplan+Schedler, Budur+Zhang, Arabello+Sacca.

Étale neighbourhood theorem

Theorem

Let $\mathcal{F}_1, \ldots, \mathcal{F}_n$ be a Σ-collection of objects in a left 2CY category $\mathcal{A}$. Let Q be a quiver with

- *Vertices $Q_0 = \{1, \ldots, n\}$.*
- *Arrows Q_1 chosen so that the number of arrows from i to j in doubled quiver $\overline{Q}$ is equal to $\mathrm{ext}^1(\mathcal{F}_i, \mathcal{F}_j)$.*

Embed $\mathbb{N}^n$ inside the coarse moduli scheme $\mathcal{M}(\mathcal{A})$ via the map sending γ to $\bigoplus_i \mathcal{F}_i^{\oplus \gamma_i}$, and inside $\mathcal{M}(\Pi_Q)$ via the map sending γ to $\bigoplus_i S_i^{\oplus \gamma_i}$. Then there is a common étale neighbourhood U of these two copies of $\mathbb{N}^n$, and a commutative diagram with Cartesian squares and étale horizontal arrows, with $\mathtt{JH}$ denoting the morphism from the respective moduli stacks

$$\begin{array}{lccccc}
\text{stacks} & \mathfrak{M}_{\mathcal{A}} & \longleftarrow\!\supset & V & \hookrightarrow & \mathfrak{M}_{\Pi_Q\text{-}mod} \\
 & \downarrow{\scriptstyle \mathtt{JH}_{\mathcal{A}}} & & \downarrow & & \downarrow{\scriptstyle \mathtt{JH}_{\Pi_Q}} \\
\text{schemes} & \mathcal{M}_{\mathcal{A}} & \longleftarrow\!\supset & U & \hookrightarrow & \mathcal{M}_{\Pi_Q\text{-}mod}
\end{array}$$

Purity for Π_Q-mod

- Given Q, the quiver $\overline{Q}$ is obtained by doubling: for every arrow $a \in Q_1$, add an arrow a^* with the opposite orientation. Then $\Pi_Q = \mathbb{C}\overline{Q}/\langle \sum_{a \in Q_1} [a, a^*] \rangle$.
- $\mathfrak{M}_\gamma(\Pi_Q)$ denotes the moduli stack of γ-dimensional Π_Q-modules, $\mathcal{M}_\gamma(\Pi_Q)$ denotes the coarse moduli space. Points of $\mathcal{M}_\gamma(\Pi_Q)$ correspond to semisimple Π_Q-modules, and at the level of points

$$\mathtt{JH}_{\Pi_Q} \colon \mathfrak{M}_\gamma(\Pi_Q) \to \mathcal{M}_\gamma(\Pi_Q)$$

takes a module to its semisimplification.

Theorem (-, 2020)

The derived direct image $\mathtt{JH}_{\Pi_Q,!}\,\mathbb{Q}_{\mathfrak{M}_\gamma(\Pi_Q)}$ is a pure MHM. In particular for some semisimple local systems $\mathcal{L}_t$ on locally closed subvarieties of $\mathcal{M}(\Pi_Q)$

$$\mathtt{JH}_{\Pi_Q,!}\,\mathbb{Q}_{\mathfrak{M}_\gamma(\Pi_Q)} \cong \bigoplus_{s \in \mathbb{Z}} \bigoplus_{t \in J_s} \mathcal{IC}(\mathcal{L}_t)[-s].$$

Some global purity results

Combining the results of the last few slides, we obtain the following

Theorem

Let $\mathtt{JH}_{\mathcal{A}} \colon \mathfrak{M}_{\mathcal{A}} \to \mathcal{M}_{\mathcal{A}}$ be the morphism from the moduli stack of objects in one of our 2CY categories to its good moduli space (in the sense of Alper). Then $\mathtt{JH}_{\mathcal{A},!}\,\mathbb{Q}$ is pure.

Some corollaries:

1. The Borel–Moore homology $\mathrm{H}^{\mathrm{BM}}(\mathfrak{M}_{\theta}^{\zeta\text{-sst}}(S), \mathbb{Q})$ is pure, if S is a smooth projective surface with $\mathcal{O}_S \cong \omega_S$. (Halpern–Leistner's conjecture).
2. The BM homology $\mathrm{H}^{\mathrm{BM}}(\mathfrak{M}_{r,d}^{\mathrm{Dol},\text{-sst}}(C), \mathbb{Q})$ is pure, as is $\mathrm{H}^{\mathrm{BM}}(\Lambda_{r,d}^{\text{-sst},\mathrm{nil}}(C), \mathbb{Q})$, the BM homology of the semistable global nilpotent cone.
4. The BM homology $\mathrm{H}(\mathfrak{M}^{\zeta\text{-sst}}(\Pi_Q), \mathbb{Q})$ is pure (recovering old result of mine).

The less perverse filtration

- It is more convenient (especially once considering Hall algebras) to consider BM homology of $\mathfrak{M}_{\mathcal{A}}$ with coefficients in the shifted constant sheaf $\tilde{\mathbb{Q}}_{\mathfrak{M}_{\mathcal{A}}} = \mathbb{Q}_{\mathfrak{M}_{\mathcal{A}}} \otimes \mathbb{L}^{\nu}$, where $\mathbb{L} = \mathsf{H}_c(\mathbb{A}^1, \mathbb{Q})$ is the Tate twist, and at a point $\mathcal{F} \in \mathrm{Ob}(\mathcal{A})$, we set $\nu = \chi(\mathcal{F}, \mathcal{F})$.
- We can write/define

$$\mathsf{H}^{\mathsf{BM}}(\mathfrak{M}_{\mathcal{A}}, \tilde{\mathbb{Q}}) = \mathbb{D}\,\mathsf{H}_c(\mathcal{M}_{\mathcal{A}}, \mathtt{JH}_{\mathcal{A},!}\,\tilde{\mathbb{Q}}_{\mathfrak{M}_{\mathcal{A}}})$$

  where $\mathbb{D}$ is Verdier duality functor (taking vector space to its dual).
- One of the consequences of the local purity theorem is that there is a natural ("less") perverse filtration defined by

$$\mathfrak{P}_{\gamma,\leq i} := \mathbb{D}\,\mathsf{H}_c\left(\mathcal{M}_{\mathcal{A}}, {}^{\mathfrak{p}}\tau^{\geq -i}\,\mathtt{JH}_{\mathcal{A},!}\,\tilde{\mathbb{Q}}_{\mathfrak{M}_{\mathcal{A}}}\right) \subset \mathsf{H}^{\mathsf{BM}}(\mathfrak{M}_{\gamma}(\Pi_Q), \mathbb{Q})$$

Preprojective CoHA and KM Lie algebras

Theorem (-,2020)

- *This perverse filtration is preserved by the Schiffmann–Vasserot product on* $\mathcal{H}_{\Pi_Q} := \bigoplus_\gamma \mathrm{H}^{\mathrm{BM}}(\mathfrak{M}_\gamma(\Pi_Q), \mathbb{Q})$.
- *The filtration begins in degree 0, and* $\mathfrak{P}_{\leq 0}\mathcal{H}_{\Pi_Q} \cong \mathrm{U}(\widetilde{\mathfrak{g}}_Q)$ *where* $\widetilde{\mathfrak{g}}_Q$ *is the BPS Lie algebra for the category of* $\Pi_Q[x]$*-modules.*
- *Considering just the zeroth cohomologically graded piece:* $\widetilde{\mathfrak{g}}_Q^0 \cong \mathfrak{n}_{Q'}$ *where* Q' *is largest full subquiver without 1-cycles, and* $\mathfrak{n}_{Q'}$ *is the Kac–Moody Lie algebra associated to the underlying graph of* Q'.
- $\sum_{i\in\mathbb{Z}} \dim(\widetilde{\mathfrak{g}}^i_{Q,\gamma})q^{-i/2} = a_\gamma(q)$ *– the Kac polynomials for* Q.

Cuspidal cohomology for Π_Q-mod

- From previous slide, $\widetilde{\mathfrak{g}}_Q$ determines the zeroth piece of the (less) perverse filtration on $\mathcal{H}_{\Pi_Q} = \mathrm{H}^{\mathrm{BM}}(\mathfrak{M}(\Pi_Q), \tilde{\mathbb{Q}})$. We'll see shortly that actually it determines all of it.
- $\widetilde{\mathfrak{g}}_Q$ is conjecturally a Borcherds algebra, i.e. determined in a precise way by its generators.

The purity theorem enables us to describe these in terms of intersection cohomology of the singular coarse moduli space $\mathcal{M}(\Pi_Q)$:

1. $\mathcal{M}(\Pi_Q)$ carries a monoidal structure, via the finite map $\oplus$ sending two Π_Q-reps to their direct sum. So $\mathrm{Perv}(\mathcal{M}(\Pi_Q))$ is monoidal: set $\mathcal{F} \overline{\otimes} \mathcal{G} := \oplus_*(\mathcal{F} \boxtimes \mathcal{G})$.
2. $\mathbb{D}\, {}^p\mathcal{H}^0\left(\mathtt{JH}_{\Pi_Q,!}\, \tilde{\mathbb{Q}}_{\mathfrak{M}_\gamma(\Pi_Q)}\right)$ is an algebra object in $\mathrm{Perv}(\mathcal{M}(\Pi_Q))$, with $\mathrm{U}(\widetilde{\mathfrak{g}}_Q)$ as its hypercohomology.
3. If there is a simple γ-dimensional Π_Q-module, $\mathcal{IC}(\tilde{\mathbb{Q}}_{\mathcal{M}_\gamma^{\mathrm{simp}}(\Pi_Q)})$ is a summand of $\mathbb{D}\, {}^p\mathcal{H}^0\left(\mathtt{JH}_{\Pi_Q,!}\, \tilde{\mathbb{Q}}_{\mathfrak{M}_\gamma(\Pi_Q)}\right)$, and so has to be primitive for support reasons. $\therefore$ $\mathrm{IC}(\mathcal{M}_\gamma(\Pi_Q)) \subset \mathcal{H}_{\Pi_Q}$ is inclusion of subspace of generators.

Cuspidal cohomology for 2CY categories

- Each of our example 2CY categories $\mathcal{A}$ carries a Hall algebra structure, local to $\mathcal{M}(\mathcal{A})$ (see Kapranov–Vasserot for $\mathrm{Coh}(S)$, Minets+ Sala–Schiffmann for Higgs sheaves, me for $\mathfrak{M}(\pi_1(\Sigma_g))$, Schiffmann–Vasserot for $\mathfrak{M}(\Pi_Q)$).
- Let $\gamma \in \mathrm{K}(\mathcal{A})$ and assume that objects $\mathcal{F}$ of class γ satisfy $\chi(\mathcal{F},\mathcal{F}) < 0$. Then by the same argument as before, there is an inclusion

$$\mathrm{IC}(\mathcal{M}_\gamma(\mathcal{A})) \subset \mathfrak{P}_{\leq 0}\mathcal{H}_{\mathcal{A},\gamma} =: \mathrm{U}(\mathfrak{g}_{\mathcal{A}})_\gamma$$

  and this subspace is primitive, in the sense that the multiplication map (restricted to $\mathrm{U}(\mathfrak{g}_{\mathcal{A}})_+$) factors through a complementary summand.
- By similar arguments there are primitive inclusions $\mathrm{IC}(\mathcal{M}_\gamma(\mathcal{A})) \subset \mathrm{U}(\mathfrak{g}_{\mathcal{A}})_{n\gamma}$ for all $n \geq 1$ if $\chi(\mathcal{F},\mathcal{F}) = 0$.

Towards global structure of $\mathrm{U}(\mathfrak{g}_{\mathcal{A}})$

Theorem

The algebra $\mathrm{U}(\mathfrak{g}_{\mathcal{A}})$ *is indeed the enveloping algebra of a BPS Lie algebra (see next slides).*

Conjecture

$\mathfrak{g}_{\mathcal{A}}$ *is a Borcherds algebra, and the inclusions*

$$\mathrm{IC}(\mathcal{M}_{\gamma}(\mathcal{A})) \subset \mathfrak{g}_{\mathcal{A},\gamma} \text{ if } \chi(\gamma,\gamma) < 0$$
$$\mathrm{IC}(\mathcal{M}_{\gamma}(\mathcal{A})) \subset \mathfrak{g}_{\mathcal{A},n\gamma} \text{ if } \chi(\gamma,\gamma) = 0$$

give all of the imaginary simple roots.

What this means for Higgs bundles

Let C be a smooth projective curve with $g \geq 2$. Set $\mathcal{A}$ to be the category of slope zero semistable Higgs bundles. We get conjectural concrete relations between E polynomials of the stack of semistable degree zero Higgs bundles and the intersection E polynomial:

1. The Borcherds algebra conjecture implies $\mathfrak{g}_{\mathcal{A}} \cong \mathrm{Free}_{\mathrm{Lie}}(\bigoplus_{r\geq 1} \mathfrak{p}_r)$ for some spaces $\mathfrak{p}_r$.
2. Generators conjecture says $\mathfrak{p}_r \cong \mathrm{IC}(\mathcal{M}^{\mathrm{Dol}}_{r,0}(C), \mathbb{Q}) \otimes \mathbb{L}^{(1-g)r^2}$.
3. Cohomological integrality theorem says that

$$\bigoplus_{r\geq 0} \mathrm{H}^{\mathrm{BM}}(\mathfrak{M}^{\mathrm{Dol}}_{r,0}(C), \tilde{\mathbb{Q}}) \otimes \mathbb{L}^{(1-g)r^2} \cong \mathrm{Sym}(\mathfrak{g}_{\mathcal{A}} \otimes \mathrm{H}(\mathrm{pt}\,/\mathbb{C}^*)).$$

Dimensional reduction

Theorem (-,2013)

Let $X \times \mathbb{A}^n$ be a smooth scheme, let f be regular function with weight one with respect to the scaling action of $\mathbb{A}^n$, let $Z = Z(\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$.

$$\pi_{X,!}\phi_f \mathbb{Q} \cong \pi_{X,!}\mathbb{Q}_Z$$
$$\mathrm{H}(X \times \mathbb{A}^n, \phi_f \mathbb{Q}) \cong \mathrm{H}^{\mathrm{BM}}(Z, \mathbb{Q}).$$

Given a quiver Q, we form the tripled quiver $\widetilde{Q}$ by adding a loop ω_i to every vertex of $\overline{Q}$. Set $W = (\sum_{i \in Q_0} \omega_i)(\sum_{a \in Q_1}[a, a^*])$. Then $\rho \mapsto \mathrm{Tr}(\rho(W))$ defines a function on $\mathfrak{M}(\mathbb{C}\widetilde{Q})$ and

$$\mathrm{H}(\mathfrak{M}_\gamma(\mathbb{C}\widetilde{Q}), \phi_{\mathrm{Tr}(W)}) \cong \mathrm{H}^{\mathrm{BM}}(\mathfrak{M}_\gamma(\Pi_Q), \mathbb{Q})$$

- An isomorphism of algebras, where the domain is the KS CoHA.
- Purity is obtained by playing the two sides of $\pi_{X,!}\phi_{\mathrm{Tr}(W)}\mathbb{Q} \cong \mathbb{Q}_{\mathfrak{M}(\Pi_Q)}$ off against each other.

The coproduct

We construct a coproduct on $\mathcal{H}_{\Pi_Q}$, following a suggestion of Kevin Costello:

- There is an identification $\mathrm{supp}(\phi_{\mathrm{Tr}(W)}) = \mathfrak{M}_\gamma(\Pi_Q[x]) \subset \mathfrak{M}_\gamma(\mathbb{C}\widetilde{Q})$; the extra loops in the tripled quiver determine the action of x.
- Let $\lambda\colon \mathfrak{M}_\gamma(\Pi_Q[x]) \to \mathrm{Sym}(\mathbb{A}^1)$ be the morphism taking a $\Pi_Q[x]$-module to the generalised eigenvalues of $x\cdot$.
- Let $U \subset \mathbb{A}^1$ be an analytic open ball, then

  $$\mathrm{H}(\mathfrak{M}_\gamma(\Pi_Q[x]), \phi_{\mathrm{Tr}(W)}) \to \mathrm{H}(\lambda^{-1}(\mathrm{Sym}(U)), \phi_{\mathrm{Tr}(W)})$$

  is an isomorphism. The assignment $U \to \mathrm{H}(\lambda^{-1}(\mathrm{Sym}(U)), \phi_{\mathrm{Tr}(W)})$ underlies a factorization cosheaf of algebras, i.e.

  $$\mathrm{H}(\lambda^{-1}(\mathrm{Sym}(\coprod_{i\in I} U_i)), \phi_{\mathrm{Tr}(W)}) \cong \bigotimes_{i\in I} \mathrm{H}(\lambda^{-1}(\mathrm{Sym}(U_i)), \phi_{\mathrm{Tr}(W)})$$

  for I indexing finitely many disjoint open sets. In particular, restriction to two disjoint open balls provides a cocommutative coproduct.
- Remark: the Poisson cobracket for $\mathcal{H}_{\Pi_Q}$ is trivial.

Integrality and geometric representation theory

- We identify $\mathbb{Q}[u] = \mathrm{H}(\mathrm{pt}/\mathbb{C}^*, \mathbb{Q})$, then $\mathbb{Q}[u]$ acts freely on each $\mathcal{H}_{\Pi_Q}$, and it's easy to check that

$$\Delta(u \cdot \alpha) = (1 \otimes u + u \otimes 1) \cdot \Delta(\alpha)$$

  and the space of primitves for Δ is a free $\mathbb{Q}[u]$-module.
- Recall that the integrality conjecture says that $[\bigoplus_{\gamma \in \mathrm{K}_0(\mathcal{A})} \mathrm{H}^{\mathrm{BM}}(\mathfrak{M}_\gamma(\mathcal{A}))] = [\mathrm{Sym}(\bigoplus_\gamma \mathrm{BPS}_{\mathcal{A},\gamma} \otimes \mathrm{H}(\mathrm{pt}/\mathbb{C}^*))]$ where the classes $[\mathrm{BPS}_{\mathcal{A},\gamma}]$ are finite and can be thought of very loosely as the cohomology of the space of primitive objects.
- Using the coproduct, we can really make sense of this for $\mathcal{A} = \Pi_Q$-mod: the Milnor–Moore theorem tells us that

$$\mathcal{H}_{\Pi_Q} \cong \mathrm{U}(\widetilde{\mathfrak{g}}_Q[u])$$

  where $\widetilde{\mathfrak{g}}_Q$ is primitive (in the coalgebraic sense).

DT theory for CY3 completions

Let $\mathcal{D}$ be one of our 2-dimensional categories, or more precisely the associated dg category. Let $\mathcal{E}$ be Keller's 3-Calabi–Yau completion

1. The category $\mathrm{Coh}(S)$ is replaced by $\mathrm{Coh}(S \times \mathbb{A}^1)$
2. The category of semistable Higgs bundles is replaced by (semistable) triples $(\mathcal{F}, f\colon \mathcal{F} \to \mathcal{F} \otimes \omega_C, g\colon \mathcal{F} \to \mathcal{F})$ with $[f, g] = 0$.
3. The category $\mathbb{C}[\pi_1(\Sigma_g)]$-mod is replaced by $\mathbb{C}[\pi_1(\Sigma_g)][x]$-mod
4. The category Π_Q-mod is replaced by $\Pi_Q[x]$-mod.

The stack of objects carries a (-1)-shifted symplectic structure (cases 1-2 need work of Brav–Dyckerhoff+Bozec–Calaque–Scherotzke). A canonical orientation is provided by the perfect $\mathcal{E}$-bimodule $\pi^*\mathcal{D}$. Thus by [BBBBJ] the stack of objects carries a perverse sheaf $\phi_{\mathcal{E}}$, and we define the cohomological DT theory to be the study of

$$\mathcal{H}_{\mathcal{E}} := \bigoplus_{\gamma \in \mathrm{K}(\mathcal{E})} \mathrm{H}(\mathfrak{M}_\gamma(\mathcal{E}), \phi_{\mathcal{E}}).$$

Cohomological integrality for CY2 categories (in progress)

Theorem (-,Kinjo)

Let $\mathcal{E}$ be a 3CY completion of a 2CY category. Then $\mathcal{H}_{\mathcal{E}}$ carries a cocommutative coproduct. The space of primitives is a free $\mathbb{Q}[u]$-module. The coproduct respects the less perverse filtration.

Theorem (-,Kinjo)

$\mathcal{H}_{\mathcal{E}}$ carries a Hall algebra structure defined via dimensional reduction. Via Milnor–Moore

$$\mathcal{H}_{\mathcal{E}} \cong \mathrm{U}(\mathfrak{g}_{\mathcal{E}}[u])$$

where $\mathfrak{g}_{\mathcal{E}}$ is the BPS Lie algebra, and $\mathfrak{g}_{\mathcal{E}}[u]$ is a Lie algebra containing it. I.e. the cohomological integrality theorem comes directly from algebra/geometric representation theory.