On the center of the small quantum group.

(joint w/ Bezrukavnikov, Shan, Vasserot).

Notation:

(simply connected)	
G semisimple group	G^L Langlands dual
$\cup$ Eg SL_n	$\cup$ PGL_n
B Borel	B^L Borel
$\cup$ upper $\triangle$ matrices	$\cup$
T max'l torus	T^L max'l torus
diagonal matrices	
$\mathfrak{g} = Lie(G)$	$\mathfrak{g}^L = Lie(G^L)$
root system	
$(X, R, X^* = \Lambda, R^*)$	(X^*, R^*, X, R)

W Weyl group

$\widetilde{W} = W \ltimes \Lambda$ affine Weyl group

Geometric Spaces

$\mathcal{K} = \mathbb{C}((t))$ $\quad \mathcal{O} = \mathbb{C}[[t]]$

Iwahori subgroup: pullback diagram

$$\begin{tikzcd} I \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & G(\mathcal{O}) \arrow[d, "t=0"] \\ B \arrow[r, hook] & G \end{tikzcd}$$

The affine flag variety $\mathcal{F}\ell$ is an ind-scheme with closed points given by $G(\mathcal{K})/I$.

It parametrizes subgroups conjugate to I.

We also have the affine Grassmannian $\mathcal{G}r$ an ind-scheme with closed points $G(\mathcal{K})/G(\mathcal{O})$.

<u>Affine Springer fibers</u>: For $\gamma \in \mathfrak{g}((t))$ we can define

$$\mathcal{F}l_\gamma = \{ gI \in \mathcal{F}l \mid \gamma \in {}^g\mathrm{Lie}(I) \}$$

We will consider this for $\gamma = ts$ for $s \in \mathfrak{g}$ regular semisimple, wlog $s \in \mathrm{Lie}(T)$.

<u>Geometry of $\mathcal{F}l_{ts}$</u>

$G/B \hookrightarrow \mathcal{F}l_{ts}$

- It is equidimensional of dimension $\dim(G/B)$.
- $T((t))$ centralizes ts, thus acts on $\mathcal{F}l_{ts}$ by left multiplication, so we get a T-action and a Λ-action

$\lambda \in \Lambda$ $\lambda: \mathbb{G}_m((t)) \to T((t))$ $\quad \Lambda \to T((t))$, $\lambda \mapsto \lambda(t)$

- Note $N(T((t)))/T(\mathcal{O}) \cong \widetilde{W} \hookrightarrow \mathcal{F}l_{ts} \hookrightarrow \mathcal{F}l$

these are precisely the T-fixed points.

<u>Applications of this affine Springer fibers</u>

1) In type A $H^*_T(\mathcal{F}l_{ts})$ is related to the Hilbert scheme of points in $\mathbb{C}^2$ $\quad \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n] \rtimes S_n$ — bimod

(upcoming joint work with Losev, Kivinen)

2) Related to the combinatorics of 2-sided cells on $\widetilde{W}$ as conjectured by Lusztig

(Yue thesis + upcoming joint work).

3) Related to the geometry of Shimura varieties

and ...

... representation theory of finite groups of Lie type (Le Hung).

4) Related to the representation theory of the small quantum group. Studied in upcoming work by Bezrukavnikov, McBreen. They prove that a regular block of $u_q\text{-mod}^{gr}$ is Koszul dual to a category of microlocal sheaves on $\mathcal{Fl}_{tS}$

5) The center of a regular block of u_q is related to $H^*(\mathcal{Fl}_{tS})$ (Upcoming joint work with Bezrukavnikov, Shan, Vasserot)

6) We can describe the T-fixed points of each component. This is related to 2) + 3)

Quantum groups

U_q be the quantum group of G^L.

Recall this is a Hopf-algebra / $\mathbb{C}(q)$ deforming the enveloping algebra.

We will consider 2 integral forms / $\mathbb{C}[q^{\pm 1}]$:

The Kac de Concini quantum group: $\mathcal{U}_q^{KdC}$ "regular enveloping alg"

The Lusztig quantum group: $\mathcal{U}_q$ "divided power enveloping alg"

We will consider these algebras at q an lth root of 1 for l an odd prime.

We have a natural map

$$\begin{tikzcd}
\mathcal{U}_q^{KdC} \arrow[rr, hook] \arrow[dr, two heads] & & \mathcal{U}_q \\
& u_q \arrow[ur] &
\end{tikzcd}$$

the image is called the small quantum group.

Note that these algebras have a grading by weights, so we can consider

$$u_q\text{-mod}^{gr}$$

the category of graded modules. Can be understood as a representation of a Harish-Chandra pair $(u_q, T^{\vee})$

Some representation theory of u_q

- The simple modules L_λ of $\mathcal{U}_q$ are given by highest wt $\lambda \in \Lambda^+$
  " " " L_λ " u_q " " " $\lambda \in \Lambda / l\Lambda$
- We have an action $\dot{l}$ of $\widetilde{W}$ on Λ given by

$$t^{\lambda} w \underset{l}{\cdot} \mu = w(\mu + \rho) - \rho + l\lambda \qquad w \in W, \quad \mu, \lambda \in \Lambda$$

$\frac{1}{2}\sum_{\alpha \in R^+} \alpha$

Linkage Principal $\operatorname{Ext}^1(L_\lambda, L_\mu) \neq 0 \Rightarrow \mu \in \widetilde{W} \underset{l}{\cdot} \lambda$.

We consider the regular block given by the Serre subcategory generated by L_λ $\lambda \in \widetilde{W}_e \cdot 0$ to get.

$$\mathcal{U}_q^{\hat{0}}\text{-mod} \quad , \quad u_q^{\hat{0}}\text{-mod}$$

Lusztig's quantum Frobenius

There is a map $Fr : \mathcal{U}_q \to \hat{\mathcal{U}}\mathfrak{g}^L$ ← completion of the enveloping alg.

$$Rep(\hat{\mathcal{U}}\mathfrak{g}^L) \cong Rep(G^L)$$

$Fr|_{u_q} = \epsilon$ counit.

$\mathcal{U}_q \curvearrowright \mathcal{U}_q$ via Hopf adjoint action

$u_q \subseteq \mathcal{U}_q$ stable under this action

Can think of u_q as a "normal sub-Hopf alg" and Fr as the quotient map.

Centers:

$$Z(\mathcal{U}_q) = \mathcal{U}_q^{\,ad\,\mathcal{U}_q} \qquad Z(u_q) = u_q^{\,ad\,u_q}$$

The adjoint action of $\mathcal{U}_q$ on u_q induces an action $\rightsquigarrow \mathcal{U}_q \curvearrowright Z(u_q)$

$G^L \curvearrowright Z(u_q)$

$$Z(\mathcal{U}_q) \cap u_q = Z(u_q)^{G^L}$$

Understanding the center

A map from $H^*(\mathcal{F}\ell)$

<u>Thm</u> (Kashiwara, Tanisaki + Kazhdan, Lusztig or Arkhipov, Bezrukavnikov, Ginzburg)

We have an equivalence

constructible along I-orbits

$$\mathcal{U}_q^{\hat{0}}\text{-mod} \cong \mathrm{Perv}_{(I)}(\mathcal{G}r)$$

<u>Thm</u> (Bezrukavnikov, Yun)

There is a Koszul duality equivalence between

Iwahori-Whittaker

$$D_{(I)}(\mathcal{G}r) \quad \text{and} \quad D_{\underline{I}W}(\mathcal{F}\ell)$$

$(\underline{I}^0, \psi)$-equiv

ψ character on $\underline{I}^0$ non-trivial on finite simple roots

<u>Lem</u> (Bezrukavnikov, BA, Shan, Vasserot (BBASV))

We have a map $H^*(\mathcal{F}\ell) \to Z(\mathcal{U}_q^{\hat{0}})$

<u>Idea of proof</u>: $H^*(\mathcal{F}\ell) \curvearrowright D(\mathcal{F}\ell)$ by natural transformations

$$\mathcal{F} \cong \mathcal{F} \otimes \mathbb{C}_{\mathcal{F}\ell} \to \mathcal{F} \otimes \mathbb{C}_{\mathcal{F}\ell}[i] \cong \mathcal{F}[i]$$

so $H^*(\mathcal{F}\ell) \curvearrowright D_{\underline{I}W}(\mathcal{F}\ell)$ and so by Koszul duality

$$H^*(\mathcal{F}\ell) \curvearrowright D_{(I)}(\mathcal{G}r)$$

$H^*(\mathcal{F}\ell)$ is pure thus this action is central ie

$\mathcal{F} \to \mathcal{F}$

$$H^*(\mathcal{F}l) \to Z(\mathcal{U}_q^{\hat{0}})$$

<u>Lem</u> (BBASV) The image of $H^*(\mathcal{F}l) \to Z(\mathcal{U}_q^{\hat{0}})$ lies in the small quantum group

ie $H^*(\mathcal{F}l) \rightsquigarrow Z(\mathcal{U}_q^{\hat{0}}) \cap U_q \cong Z(U_q^{\hat{0}})^{G^L}$.

$U_q\text{-mod} \cong \mathcal{O}_{G^L}\text{-mod}(\mathcal{U}_q\text{-mod})$

$\mathcal{U}_q\text{-mod} \circlearrowleft \mathrm{Rep}(G^L) \ni \mathcal{O}_{G^L}\text{-mod}$.

A map from $H^*(\mathcal{F}l_{+s})$

<u>Thm</u> (Goresky, Kottwitz and MacPherson)
Under some assumptions, the image of

$$H^*_T(X) \hookrightarrow H^*_T(X^T)$$

can be determined by understanding 1-dim T-orbits.

<u>Rmk</u>: This gives a description of $H^*_T(\mathcal{F}l_{+S})$ in terms of the T-fixed points $\widetilde{W}$ and the 1-dim T-orbits

We have a central deformation $\mathcal{U}_q^{KdC,t}$ of $\mathcal{U}_q$

over $\mathbb{C}[\mathrm{Lie}(T^L)]$, with maps

$$\mathcal{U}_q^{kdC} \twoheadrightarrow \mathcal{U}_q^{kdC,t} \twoheadrightarrow u_q$$

Z_ℓ ℓ-center

Prop (BBASV)

$$Z(\mathcal{U}_q^{kdC,t,\hat{0}}\text{-mod}^{gr}) \cong H_T^*(\mathcal{F}\ell_{tS})$$

Idea of proof: We can understand the center of $\mathcal{U}_q^{kdC,t,\hat{0}}\text{-mod}^{gr}$ at the generic point and codim 1 points in $\mathrm{Lie}(T^L)$

$\hookrightarrow$ Rep thy is related to $T^L \rightsquigarrow$ fixed pts

$\longrightarrow$ Rep thy related to $SL_2 \rightsquigarrow$ q-duality

This description matches GKM description of $H_T^*(\mathcal{F}\ell_{tS})$

Cor
$$H^*(\mathcal{F}\ell_{tS}) \hookrightarrow Z(u_q^{\hat{0}}\text{-mod}^{gr})$$
$$H^*(\mathcal{F}\ell_{tS})^\wedge \hookrightarrow Z(u_q^{\hat{0}})^{T^L} \qquad (u_q, T^L)$$

Compatibility of the maps

Thm (BBASV) We have a commutative diagram

$$\begin{tikzcd}
H^*(\mathcal{F}\ell) \arrow[r] \arrow[d] & Z(u_q^{\hat{0}})^{G^L} \arrow[d, "\wr"] \\
H^*(\mathcal{F}\ell_{tS})^\wedge \arrow[r, hook] & Z(u_q^{\hat{0}})^{T^L}
\end{tikzcd}$$

Lem/Conj $\mathrm{Im}(H^*(\mathcal{F}\ell) \to H^*(\mathcal{F}\ell_{tS})^\wedge) \cong H^*(\mathcal{F}\ell_{tS})^{\widetilde{W}}$

Proven in Type A, Conjecture in general.

Conj:

$$\boxed{\begin{tikzcd} H^*(\mathcal{F}l_{ts})^{\widetilde{W}} \arrow[r, phantom, "\cong"] \arrow[d, hook] & Z(u_q\hat{o})^{G^L} \arrow[d, hook] \\ H^*(\mathcal{F}l_{ts})^{\Lambda} \arrow[r, phantom, "\cong"] & Z(u_q\hat{o})^{T^L} \end{tikzcd}}$$

equality here $\Rightarrow G^L \curvearrowright Z(u_q^{\hat{o}})$ is trivial.

In type A this

In type A this is an equality:

- joint with Losev
- joint w/ Bezrukavnikov, Shan, Vasserot

Both depend on Haiman proof of $n!$ conj.

In fact by proof with Losev

it is equivalent.

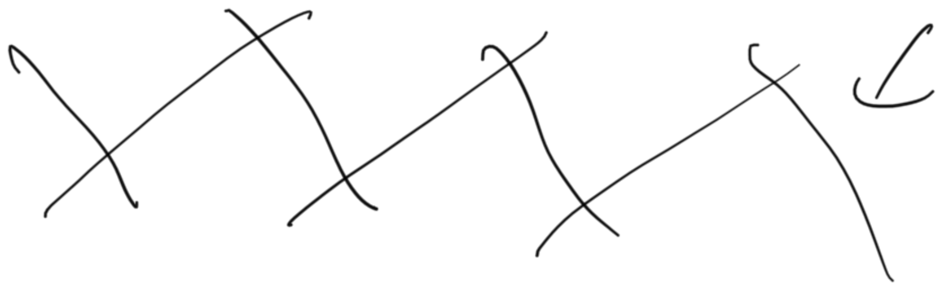
Expect: $\dim H^*(\mathcal{F}l_{ts})^{\widetilde{W}} = (h+1)^{rk}$

$\qquad\qquad\qquad\qquad\qquad\qquad \shortparallel$

in type A $\dim H^*(\mathcal{F}l \ldots)^{\Lambda} \ldots$

$\dim H^*(\mathcal{F}l_{ts}) \underset{\text{conj}}{=} \dim Z(U_q^{\hat 0})$.

Eg SL_2.

$\mathcal{F}l_{ts}$  $\mathbb{P}^1$.

$U_q^{\hat 0}$-mod^{gr}

$$\begin{tikzcd} \cdots \bullet \arrow[r, shift left, "\varphi_n^+"] & \bullet \arrow[l, shift left, "\varphi_n^-"] \arrow[r, shift left, "\varphi_{n+1}^+"] & \bullet \arrow[l, shift left, "\varphi_{n+1}^-"] \cdots \end{tikzcd}$$

$$\varphi_{n+1}^+ \circ \varphi_n^+ = 0$$

$$\varphi_n^- \circ \varphi_{n+1}^- = 0$$

$$\varphi_n^+ \circ \varphi_n^- = \varphi_{n+1}^- \circ \varphi_{n+1}^+ = \Omega_n$$

$$Z = \langle 1, \Omega_n \rangle . \quad \Omega_n \cdot \Omega_k = 0$$

$H^*($  $) = \langle 1, c_n \rangle$ $\leftarrow$ for each $\mathbb{P}^1$.

$c_n c_n = 0.$

$$H_T^*(\mathcal{Fl}_{tS}) \hookrightarrow \prod_{\tilde{W}} \underset{H_T^*(pt)}{\mathbb{C}[t]}$$

$$(f_w)$$

$$\alpha \mid f_w - f_{s_{\alpha,k} w}$$

$$\mathbb{C}[x_1, \ldots x_n, y_1, \ldots y_n] \rtimes S_n$$

$$\left(H^*(\mathcal{Fl}_{tS})^\wedge\right)^* \cong \mathbb{C}[x_1 \ldots y_1 \ldots y_n] \Big/ (x_1 \ldots x_n, y_1 \ldots y_n)^W$$

the

$$H_*(\mathcal{Fl}_{tS})^\wedge$$

$$H_*^{BMT}(\mathcal{Fl}_{tS})$$

- equiv parameters, translations, monodromy,
- chern classes, Springer action