Cluster structures on subvarieties of the Grassmannian

Melissa Sherman-Bennett (UC Berkeley)

joint work with C. Fraser (arXiv:2006.10247)
M. Parisi and L. Williams (arXiv:2104.08254)

Davis Algebraic Geometry Seminar

The setting

Fix integers $0 < k < n$.

- $Gr_{k,n} := \{V \subseteq \mathbb{C}^n : \dim(V) = k\}$
- $V \in Gr_{k,n} \rightsquigarrow$ full rank $k \times n$ matrix A whose rows span V
- $I \subset \{1, \ldots, n\}$ with $|I| = k$. *Plücker coordinate* $P_I(V) =$ max'l minor of A located in column set I.
- The *Schubert cell*

$$\Omega_I := \{V \in Gr_{k,n} : P_I \text{ lex smallest nonzero Plücker}\}.$$

The *open Schubert variety*

$$X_I^\circ := \Omega_I \setminus \{V \in \Omega_I : P_I P_{I_2} \cdots P_{I_n} = 0\}.$$

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- Knutson-Lam-Speyer (Lusztig, Postnikov, Rietsch): *open postroid variety*

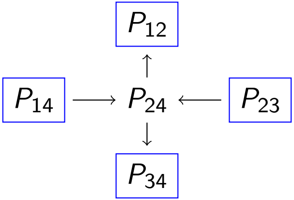
$$\begin{aligned} \Pi_\mu^\circ &= \text{intersection of } n \text{ cyclically shifted Schubert cells} \\ &= \text{proj. of open Richardson variety } B^- vB/B \cap BwB/B \subset Fl_n \end{aligned}$$

$$\mathbb{C}[\Pi_\mu] = \mathbb{C}[Gr_{k,n}]/\langle \text{some } P_I \rangle$$

Cluster algebras, briefly

Start with V affine.

- Initial *seed* $\Sigma \subset \mathbb{C}[V]$ of *cluster variables* labeling a directed graph.



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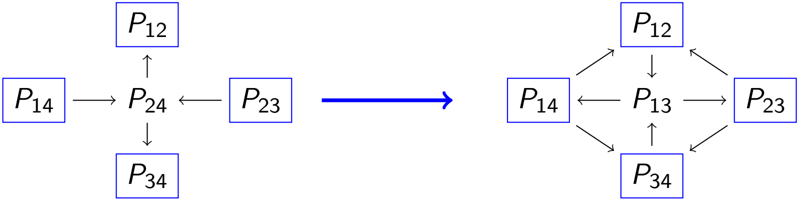

- *Mutation*: local move to get new seed. Cluster variable x is exchanged for new cluster variable x' satisfying $x \cdot x' = A + B$.
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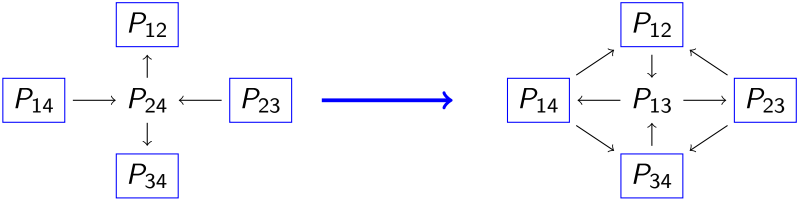

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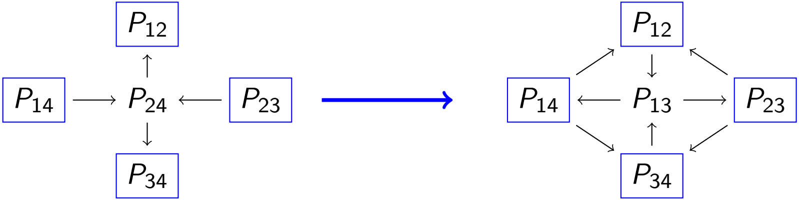

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Σ gives a *cluster structure* for V if $\mathbb{C}[V] = \mathcal{A}(\Sigma)$.

More geometrically

- Each seed Σ defines a *cluster torus* $T_\Sigma \subset V$ where cluster variables of Σ are non-vanishing.
- Cluster tori are glued according to rational mutation maps.
- Σ gives a cluster structure for $V \implies V$ is (up to codimension 2)

$$\bigcup_{\Sigma'} T_{\Sigma'}$$

  where the union is over seeds Σ' obtained from Σ by any sequence of mutations.
- *Totally positive part* $V^{>0}$: where all cluster variables are positive.

Cluster algebras, briefly

- If $\mathbb{C}[V] = \mathcal{A}(\Sigma)$, then $\mathbb{C}[V]$ has a vector space basis of *theta functions* with positive structure constants (Gross–Hacking–Keel–Kontsevich).

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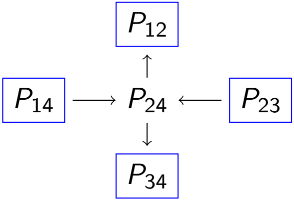

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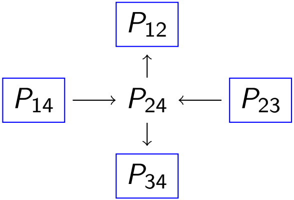

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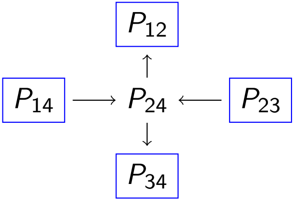

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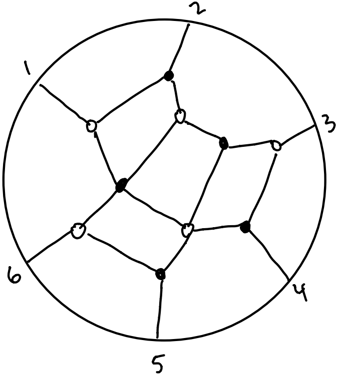
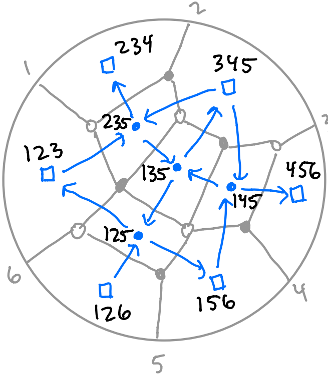
Problem: Give explicit descriptions of cluster monomials (equiv. seeds) in $\mathbb{C}[V]$.

History

Theorem (Scott '06)

$\mathbb{C}[Gr^\circ_{k,n}]$ *is a cluster algebra and Postnikov's plabic graphs for* $Gr^\circ_{k,n}$ *give seeds (consisting entirely of Plücker coordinates).*





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Theorem (Galashin–Lam '19)

$\mathbb{C}[\Pi^\circ_\mu]$ is a cluster algebra and plabic graphs for Π°_μ give seeds in this cluster algebra.

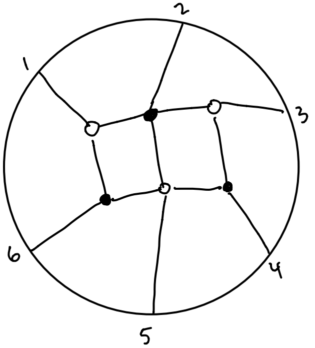
- (Leclerc '16): Coordinate rings of open Richardson varieties in Fl_n have a cluster subalgebra $\implies$ $\mathbb{C}[\Pi^\circ_\mu]$ has a cluster subalgebra.
- (Serhiyenko–SB–Williams '19): For X°_I, plabic graphs give seeds in Leclerc's cluster algebra.
- (Galashin–Lam '19): For Π°_μ, plabic graphs give seeds in Leclerc's cluster (sub)algebra and Leclerc's subalgebra equals $\mathbb{C}[\Pi^\circ_\mu]$.

A bit more on seeds from plabic graphs

A *plabic graph* G of type (k, n): planar, embedded in disk, boundary vertices $1, \dots, n$ going clockwise, internal vertices colored black and white.

To get a seed Σ_G:

- Directed graph is dual graph (boundary faces are frozen).



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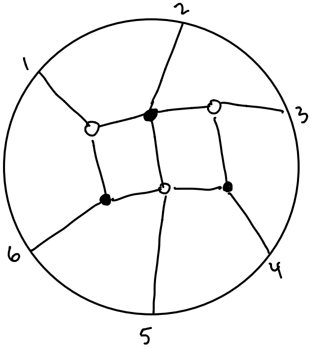
  Every face labeled by k-elt subset, which we interpret as Plücker coordinate.



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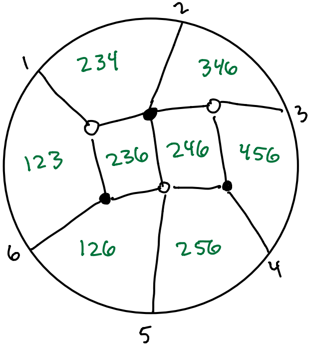

Trip permutation μ tells you which positroid variety G is a plabic graph for. All seeds Σ_G, where G has trip permutation μ, are related by mutation.

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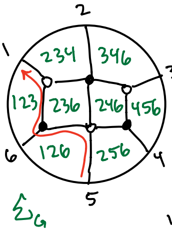

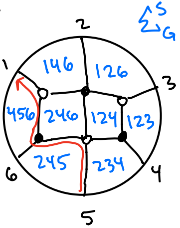

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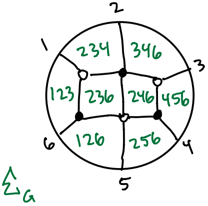
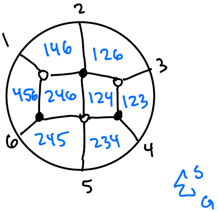
A closer look at cluster structure for Π_μ°

1. Many nonzero P_I are not cluster variables in $\mathcal{A}(\Sigma_G)$.
2. $\exists$ seeds whose cluster variables are $P_I \cdot$(*Laurent mono. in frozens*)... combinatorial source?
3. No sequence of mutations between Σ_G^S and Σ_G! Two convention choices give different cluster algebras $\mathcal{A}(\Sigma_G)$ and $\mathcal{A}(\Sigma_G^S)$.

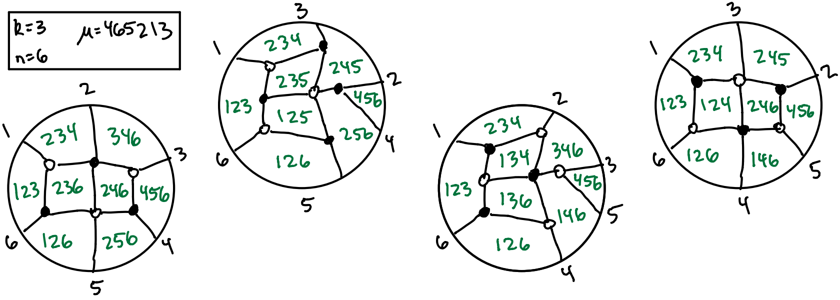




Main result

Theorem (Fraser–SB '20)

$\mathbb{C}[\Pi^\circ_\mu]$ *can be identified with many different cluster algebras (with different frozen and cluster variables), whose seeds are given by certain relabeled plabic graphs.*



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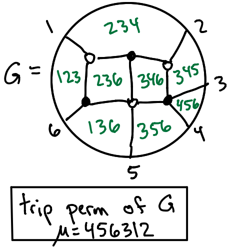
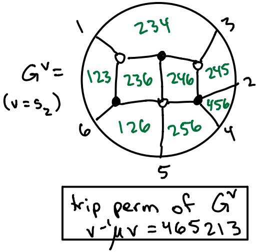
Takeaway: Relabeled plabic graphs give many additional explicit seeds for $\mathbb{C}[\Pi^\circ_\mu]$, with cluster variables $P_I\cdot$(Laurent mono. in frozens).

Relabeled plabic graphs

G a plabic graph of type (k, n), $v \in S_n$. The *relabeled plabic graph* G^v is obtained from G by applying v to its boundary vertex labels.





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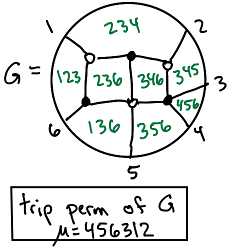

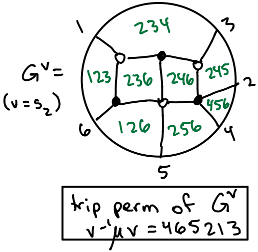

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Note: The "source" seed Σ_G^S is the same as $\Sigma_{G^{(\mu^{-1})}}$.

Main results, more precisely

Theorem (Fraser–SB '20)

Let G^v be a relabeled plabic graph with trip permutation μ where $\mu v \leq \mu$ in the circular weak order. The following are equivalent.

1. $\mathcal{A}(\Sigma_{G^v}) = \mathbb{C}[\Pi_\mu^\circ]$.
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Proving (2) $\implies$ (1) involves showing $\Pi^\circ_{v\mu v^{-1}} \cong \Pi^\circ_\mu$ using a permuted version of the Muller-Speyer twist map τ. The permuted double twist $\tau^2 \circ v$ sends Σ^S_G to Σ_{G^v} (up to frozens).

Relationship of relabeled plabic graph seeds?

Conjecture

H^w, G plabic graphs with trip permutation μ such that $\mathcal{A}(\Sigma_{H^w}) = \mathbb{C}[\Pi^\circ_\mu]$. Then Σ_{H^w} and Σ_G are related by mutations followed by "nice" rescaling by Laurent monomials in frozens.

If conjecture holds, then Σ_{H^w} is, up to frozens, a seed in $\mathcal{A}(\Sigma_G)$. In particular, cluster monomials from Σ_{H^w} are cluster monomials for $\mathcal{A}(\Sigma_G)$.

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Theorem (Fraser–SB '20)

Conjecture holds for open Schubert varieties X_I°.

Partial results for arbitrary positroid varieties, including that all relabeled plabic graph cluster structures give the same positive part of Π_μ°.

Some questions

- Let Σ be a seed in target cluster structure on $\mathbb{C}[\Pi^\circ_\mu]$ with cluster variables $\{P_I \cdot (\text{Laurent mono. in frozens})\}$. Does Σ come from a relabeled plabic graph?
- Are all Plücker coordinates in $\mathbb{C}[\Pi^\circ_\mu]$ cluster monomials? From a relabeled plabic graph seed?

The amplituhedron

$Z = n \times (k+2)$ matrix with positive Plücker coordinates.

$$\begin{array}{ccccc} & & \tilde{Z} & & \\ Gr_{k,n} & \dashrightarrow & Gr_{k,k+2} & \hookrightarrow & Gr_{2,n} \\ [C] & \longmapsto & [CZ] & & \\ & & [Y] & \longmapsto & [Y^{\perp} Z^{t}] \end{array}$$

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- Totally nonnegative Grassmannian

$$Gr_{k,n}^{\geq 0} = \bigcup_{\mu} \Pi_{\mu}^{\circ\ >0}$$

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- Totally nonnegative Grassmannian

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- $m = 2$ amplituhedron

$$\mathcal{A}_{n,k,2} = \tilde{Z}(Gr_{k,n}^{\geq 0}) = \bigcup_{\mu} \tilde{Z}(\Pi_{\mu}^{\circ\ >0}).$$

Decompositions of the amplituhedron

Dream: Can we find set $\mathcal{M}$ of decorated permutations so that

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Current state: Can do "top-dimension" part. Can find many sets $\mathcal{M}$ of $2k$-dimensional positroid varieties where

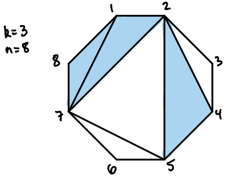
$$\mathcal{A}_{n,k,2} = \overline{\bigcup_{\mu \in \mathcal{M}} \tilde{Z}(\Pi_\mu^{\circ\ >0})}$$

s.t. $\tilde{Z}(\Pi_\mu^{\circ\ >0})$ are disjoint, homeo. to open balls of dimension $2k$, and equal to $V_\mu^{>0}$.

Cluster structures related to the amplituhedron

Results from [Parisi–SB–Williams '21]:
Choose a "nice" $2k$-dimensional Π_μ° and let $V_\mu := \tilde{Z}(\bigcup_G T_{\Sigma_G})$.

- V_μ is a cluster variety and $V_\mu^{>0} = \tilde{Z}(\Pi_\mu^{\circ\ >0})$.
- Cluster variables are signed (ratios of) Plücker coordinates.
- Combinatorial object encoding seeds: bicolored triangulations.



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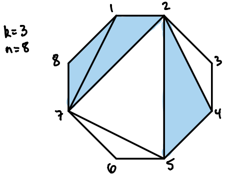

Also give many decompositions

$$\mathcal{A}_{n,k,2} = \overline{\bigcup_{\mu \in \mathcal{M}} V_\mu^{>0}}$$

with $V_\mu^{>0}$ pairwise-disjoint.

Thanks for listening!

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